

The Fast Fourier Transform Demystified

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1. INTRODUCTION

What is the Fourier Transform, anyway? And how does a nasty integral find its way into the electronics world? These questions bothered me greatly. Quite possibly, these same questions are bothering you, too.

2. BACKGROUND

If you needed to know the relative strength of a signal in a certain frequency range, you could build a filter and measure the power that passes through. But what if you wanted to analyze more than one frequency band? Would you build hundreds of filters and hope that the components are accurate enough to get the signals you want? An easier method of decomposing a signal into its components involves using a modification of the work of a mathematician named Fourier. Fourier's work is to an electrical signal what an optical prism is to light.

In general, the Fourier transform is used to move a function from amplitude as a function of time to amplitude as a function of frequency. Looking at a function which describes amplitude in terms of frequency can reveal the signal strength in a particular range of frequencies.

As it stands, the Fourier Transform is a bit hard to do with discrete data. The equation

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi f t} dt$$

applies best to continuous functions. (In this equation, $x(t)$ is the function in time, and $X(f)$ is the corresponding frequency function. f_t is the frequency to analyze.)

It was reasoned that someone would have to come up with a discrete transform to deal with signals sampled at discrete intervals. They develop the **DFT**, and this serves the needs of all two people who worked with discrete data at that early time in history. It has but one drawback: it is slow. The transform essentially multiplies each sample by a sine wave and a cosine wave (a sine wave 90° out of phase) for each frequency to be analyzed.

So, some junior programmers at IBM decide that they have too much time, and they proceed to develop a faster method. Nobody at their lab (the Thomas J. Watson Research Park) had shown them their official projects, so they decided to have some mathematical fun. Their work paid off, and the FFT most commonly used is named in

their honor (the Cooley-Tukey algorithm).

3. DETAILS

3.1. The DFT

The reason that compelled Cooley and Tukey to take such drastic action is that a DFT has inherent redundancies. In the DFT, you have a set of discrete samples. Let's say, for purposes of explanation, that it has 1024 points, and that point n is denoted by $f(n)$. You then pick a frequency to analyze against the sample. (There are certain restrictions, mentioned below, that apply to the selected frequency.) You would then generate a sine wave at that frequency, and a cosine (sine + 90 degrees phase shift) wave. Call them $s(n)$ and $c(n)$, respectively. You then calculate: $X(m) = \frac{1}{n} \sum_{k=0}^{n-1} x(k)W^{mk}$ where $W = e^{\frac{-j2\pi}{n}}$. Raising e to complex powers gets hairy, however. It will be shown later that the whole mess

is almost equivalent to $\frac{1}{n} \sqrt{\left(\sum_{i=1}^n f(n)s(n)\right)^2 + \left(\sum_{i=1}^n f(n)c(n)\right)^2}$. The multiplication corresponds to modulation of signals, which is used in AM radio for both transmission (multiplying a carrier by an audio signal) and for receiving (when the local oscillator wants to generate sum and difference waves to filter out the carrier).

The preceding equation, in English, means that you calculate the sum of all the products of the function at point n and the sine wave at point n . Then, you calculate the sum of the function at point n and the cosine wave at point n . You square both sums, add the resulting numbers, and take the square root of that. (Wow. I'm glad that a computer usually does all that...) You now have a relative frequency strength for f . After all, in many cases, all you are interested in is the magnitude of the complex number (denoted $X(f)$ above) that the transform returns. The relative strength of frequency f is what $X(f)$ is meant to denote, and the angle that $X(f)$ forms in the complex plane is the phase shift that a sine wave would go through were it to reconstruct the original sampled wave.

The problem is that this is an "order N^2 " operation, meaning that there are two variables involved (number of frequencies to analyze and number of samples) and the computation time is proportional to the product of these two variables.

3.2. The FFT

3.2.1. The Algorithm

Cooley and Tukey noticed that if you do this for a number of frequencies, you get many redundancies in the points which you calculate. First of all, the sine and cosine waves are the same waves at different positions. One could get away with calculating just the sine wave, and shifting the phase to get the cosine wave. Also, one can use the symmetry of the waves to even further reduce calculation time. Plus, the waves, as the frequency being checked increases, overlap more and more. They devised a method which involves very little computation, and runs approximately 200 times faster than a DFT on the same sample wave. This is an "order $N \log_2 N$ " operation, meaning that for large numbers of samples or frequencies to analyze, the computation time rises nearly linearly (as opposed to skyrocketing with a straight DFT).

In the complex-number variation of the equation, there is a term known as “W”. It is computed as follows:

$$W = e^{\frac{-j2\pi}{n}}$$

In each of the circles in the tier diagram, there is a number, p . This is the power to which the “W” term must be raised. Your math book probably uses i , but i represents current in electronics. W is calculated using Euler’s formula, which says that $e^{-jx} = \cos(x) + j\sin(x)$. (see the Math Review at the end). The dashed line in the tier diagram indicates the x value for the first term of the following equation, and the solid line indicates the x for the second term. Here is the operation which happens in each circle:
 $x(m_2) + W^p x(m_2)$

3.2.2. The Results

Given n input points, the FFT returns n frequency components, of which the first $\frac{n}{2}$ are valid. (The other components are mirror images of the first n and are considered invalid since the frequencies they represent do not satisfy the Nyquist Theorem below.) They start with the DC component, and are spaced apart by a frequency of $\frac{1}{n\Delta t}$. The magnitude of the complex number returned is the frequency’s relative strength.

3.2.3. Caveats

The Nyquist Theorem specifies that you must sample at a frequency **over** twice the maximum frequency you want to analyze. That means that for each cycle of the input wave, the average number of samples must be over two. This theorem explains why compact discs are recorded with a 44 KHz sampling frequency even though the human ear can only hear sounds up to about 20 KHz (your mileage may vary). If the Nyquist Theorem’s conditions are not satisfied, a phenomenon known as aliasing occurs. In the case of a CD, there would not be enough samples to fully define each waveform at a high fundamental frequency, and the sound would be much lower than expected.

4. Math Review

The justification behind the expansion of the e^x term is as follows. Most transcendental functions (including sine, cosine and e^x) can be represented as an infinite sum of polynomial terms. The derivation can be found in any reputable Calculus text under the heading of “Taylor Polynomials”. These sums work when complex numbers are involved. For instance:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \dots$$

$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$$

$$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$$

Letting $x = i\theta$, Euler’s formula becomes obvious to the most casual observer:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \dots \\ &= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} + \dots \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right) \end{aligned}$$

which looks surprisingly like

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$