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INTRODUCTION

Notwithstanding its title, the reader will not find in this book a systematic account of this huge subject. Certain classical aspects have been passed by, and the true title ought to be "Various questions of elementary combinatorial analysis". For instance, we only touch upon the subject of graphs and configurations, but there exists a very extensive and good literature on this subject. For this we refer the reader to the bibliography at the end of the volume.

The true beginnings of combinatorial analysis (also called combinatorial analysis) coincide with the beginnings of probability theory in the 17th century. For about two centuries it vanished as an autonomous subject. But the advance of statistics, with an ever-increasing demand for configurations as well as the advent and development of computers, have, beyond doubt, contributed to reinstating this subject after such a long period of negligence.

For a long time the aim of combinatorial analysis was to count the different ways of arranging objects under given circumstances. Hence, many of the traditional problems of analysis or geometry which are concerned at a certain moment with finite structures, have a combinatorial character. Today, combinatorial analysis is also relevant to problems of existence, estimation and structuration, like all other parts of mathematics, but exclusively for finite sets.

My idea is here to take the uninitiated reader along a path strewn with particular problems, and I can very well imagine that this journey may jolt a student who is used to easy generalizations, especially when only some of the questions I treat can be extended at all, and difficult or unsolved extensions at that, too. Meanwhile, the treatise remains firmly elementary and almost no mathematics of advanced college level will be necessary.

At the end of each chapter I provide statements in the form of exercises that serve as supplementary material, and I have indicated with a star those that seem most difficult. In this respect, I have attempted to write down
these 219 questions with their \textit{answers}, so they can be consulted as a kind of \textit{compendium}.

The first items I should quote and recommend from the bibliography are the three great classical treatises of Netto, MacMahon and Riordan. The bibliographical references, all between brackets, indicate the author's name and the year of publication. Thus, [Abel, 1826] refers, in the \textit{bibliography of articles}, to the paper by Abel, published in 1826. Books are indicated by a star. So, for instance, [*Riordan, 1968] refers, in the \textit{bibliography of books}, to the book by Riordan, published in 1968. Suffixes a, b, c, distinguish, for the same author, different articles that appeared in the same year.

Each chapter is virtually independent of the others, except of the first; but the use of the index will make it easy to consult each part of the book separately.

I have taken the opportunity in this English edition to correct some printing errors and to improve certain points, taking into account the suggestions which several readers kindly communicated to me and to whom I feel indebted and most grateful.

\textbf{SYMBOLS AND ABBREVIATIONS}

- $\Psi_k(N)$ set of $k$-arrangements of $N$
- $B_{n,k}$ partial Bell polynomials
- $C$ set of complex numbers
- $E(X)$ expectation of random variable $X$
- $GF$ generating function
- $N$ set of integers $\geq 0$
- $P(A)$ probability of event $A$
- $\Psi(N)$ set of subsets of $N$
- $\Psi(N)$ set of nonempty subsets of $N$
- $\Psi_k(N)$ set of subsets of $N$ containing $k$ elements
- $A \cup B = A \cap B$, understanding that $A \cap B = \emptyset$
- $R$ set of real numbers
- $RV$ random variable
- $Z$ set of all integers $\geq 0$
- $\Delta$ difference operator
- $\#$ indicates beginning and end of the proof of a theorem
- $:=$ equals by definition
- $[n]$ the set $\{1, 2, 3, \ldots, n\}$ of the first $n$ positive integers
- $n!$ $n$ factorial $= \text{the product } 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$
- $(x)_k = x(x-1)\ldots(x-k+1)$
- $\langle x\rangle_k = x(x+1)\ldots(x+k-1)$
- $[x]$ the greatest integer less than or equal to $x$
- $\|x\|$ the nearest integer to $x$
- $\binom{n}{k}$ binomial coefficient $= \frac{(n)_k}{k!}$
- $s(n, k)$ Stirling number of the first kind
- $S(n, k)$ Stirling number of the second kind
- $|N|$ number of elements of set $N$
- $k$ bound variable, with dot underneath
- $\overline{A}$, $\overline{A}$ complement of subset $A$
- $C_{n,f}$ coefficient of $t^n$ in the formal series $f$
- $\{x \mid \mathcal{P}\}$ set of all $x$ with property $\mathcal{P}$
- $N^M$ set of maps of $M$ into $N$
VOCABULARY OF COMBINATORIAL ANALYSIS

In this chapter we define the language we will use and we introduce those elementary concepts which will be referred to throughout the book. As much as possible, the chosen notations will not be new; we will use only those that actually occur in publications. We will not be afraid to use two different symbols for the same thing, as one may be preferable to the other, depending on circumstances. Thus, for example, $\mathcal{A}$ and $\mathcal{C}$ both denote the complement of $A$, $A \cap B$ and $AB$ stand for the intersection of $A$ and $B$, etc. For the rest, it seems desirable to avoid taking positions and to obtain the flexibility which is necessary to be able to read different authors.

1.1. SUBSETS OF A SET; OPERATIONS

In the following we suppose the reader to be familiar with the rudiments of set theory, in the naive sense, as they are taught in any introductory mathematics course. This section just defines the notations.

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ denote the set of the non-negative integers including zero, the set of all integers $\leq 0$, the set of the real numbers and the set of the complex numbers, respectively.

We will sometimes use the following logical abbreviations: $\exists$ (=there exists at least one), $\forall$ (=for all), $\Rightarrow$ (=implies), $\Leftarrow$ (=if), $\iff$ (=if and only if).

When a set $\Omega$ and one of its elements $\omega$ is given, we write “$\omega \in \Omega$” and we say “$\omega$ is element of $\Omega$” or also “$\omega$ belongs to $\Omega$” or “$\omega$ in $\Omega$”.

Let $\Pi$ be the subset of elements $\omega$ of $\Omega$ that have a certain property $\mathcal{P}$, $\Pi \subseteq \Omega$, then we denote this by:

$$[1a] \quad \Pi := \{ \omega \mid \omega \in \Omega, \mathcal{P} \},$$

and we say this as follows: “$\Pi$ equals by definition the set of elements $\omega$ of $\Omega$ satisfying $\mathcal{P}$”. When the list of elements $a, b, c, \ldots, l$ that constitute
ADVANCED COMBINATORICS VOCABULARY OF COMBINATORIAL ANALYSIS

If \( N \) is a finite set, \(|N|\) denotes the number of its elements. Hence \(|N| = \text{card } N\), the cardinal of \( N \), also denoted by \( \overline{N} \).

\( \mathcal{P}(N) \) is the set of all subsets of \( N \), including the empty set; \( \mathcal{P}'(N) \) denotes the set of all nonempty subsets, or combinations, or blocks, of \( N \); hence, when \( A \) is a subset of \( N \), we will denote this by \( A \subset N \) or by \( A \in \mathcal{P}(N) \), as we like. For \( A, B \) subsets of \( N \), we recall that

\[
A \cap B := \{ x \mid x \in A \land x \in B \},
\]

\[
A \cup B := \{ x \mid x \in A \lor x \in B \}.
\]

The (set theoretic) difference of two subsets \( A \) and \( B \) of \( N \) is defined by:

\[
A \setminus B := \{ x \mid x \in A \land x \notin B \}.
\]

The complement of \( A \subset N \) is the subset \( N \setminus A \) of \( N \), also denoted by \( \overline{A} \), or \( \complement A \), or \( ^c A \). The operation which assigns to \( A \) the set \( \overline{A} \) is called \( \complement \)-operation:

\[
[1c] \quad A \setminus B = A \cap \overline{B}.
\]

\( \mathcal{P}(N) \) is made into a Boolean algebra by the operations \( \cup \), \( \cap \) and \( \complement \). Such a structure consists of a certain set \( M \) (here \( \mathcal{P}(N) \)) with two operations \( \cup \) and \( \cap \) (here \( \cup = \cup \), \( \cap = \cap \)) and a map of \( M \) into itself: \( a \rightarrow \bar{a} \) (here \( A \rightarrow \complement A = \mathcal{C} A \)) such that for all \( a, b, c, \ldots \in M \), we have:

\[
[1d] \quad (I) \quad (a \lor b) \lor c = a \lor (b \lor c),
\]

\[
[1d] \quad (II) \quad (a \land b) \land c = a \land (b \land c) \quad \text{(associativity of } \lor \text{ and } \land). \]

\[
[1d] \quad (III) \quad a \lor b = b \lor a,
\]

\[
[1d] \quad (IV) \quad a \land b = b \land a \quad \text{(commutativity of } \lor \text{ and } \land). \]

\[
[1d] \quad (V) \quad \text{There exists a (unique) neutral element denoted by } 0, \text{ for } \lor: a \lor 0 = 0 \lor a = a.
\]

The most important interrelations between the operations \( \cup \), \( \cap \), \( \complement \) are the following:

**DE Morgan Formulas.** Let \((A_i)_{i \in I} \) and \((B_k)_{k \in K} \) be two families of \( N \), \( A_i \subset N, B_k \subset N, i \in I, k \in K \). Then:

\[
[1e] \quad \complement (\bigcup_{i \in I} A_i) = \bigcap_{i \in I} (\complement A_i)
\]

\[
[1f] \quad \complement (\bigcap_{i \in I} A_i) = \bigcup_{i \in I} (\complement A_i)
\]

\[
[1g] \quad (\bigcup_{i \in I} A_i) \cap (\bigcup_{k \in K} B_k) = \bigcup_{i \in I} (A_i \cap B_k)
\]

\[
[1h] \quad (\bigcap_{i \in I} A_i) \cup (\bigcap_{k \in K} B_k) = \bigcap_{k \in K} (A_i \cup B_k)
\]

A system \( \mathcal{S} \) of \( N \) is a nonempty (unordered) set of blocks of \( N \), without repetition (\( \Rightarrow \mathcal{S} \in \mathcal{P}'(\mathcal{P}'(N)) \)); a \( k \)-system is a system consisting of \( k \) blocks.

**1.2. Product sets**

Let be given \( m \) finite sets \( N_i, 1 \leq i \leq m \), and recall that the product set \( \prod_{i=1}^m N_i \) or Cartesian product of the \( N_i \) is the set of the \( m \)-tuples \( \langle y \rangle = \langle y_1, y_2, \ldots, y_m \rangle \), where \( y_i \in N_i \) for all \( i = 1, 2, \ldots, m \). The product set is also denoted by \( N_1 \times N_2 \times \cdots \times N_m \) or by \( N_1N_2N_3 \cdots N_m \) if there is no danger for confusion. We call \( y_i \) the projection of \( \langle y \rangle \) on \( N_i \), denoted by \( p_i(y) \).

If \( N_1 = N_2 = \cdots = N_m = N \), the product set is also denoted by \( N^m \); the diagonal \( \Delta \) of \( N^m \) is hence the set of the \( m \)-tuples such that \( y_1 = y_2 = \cdots = y_m \).

**Theorem.** The number of elements of the product set of a finite number.
of finite sets satisfies:

\[ [2a] \quad \prod_{i=1}^{m} |N_i| = \prod_{i=1}^{m} |N_i| = |N_1| \cdot |N_2| \cdots |N_m| . \]

In fact, the number of \( m \)-tuples \((y_1, y_2, \ldots, y_m)\) is equal to the product of the number of possible choices for \( y_1 \) in \( N_1 \), which is \(|N_1|\), by the number of possible choices of \( y_2 \) in \( N_2 \), which is \(|N_2|\), etc., by the number of possible choices of \( y_m \) in \( N_m \), which is \(|N_m|\), because these choices can be done independently from each other. \( \square \)

**Example.** What is the number \( d(n) \) of factors of \( n \), with prime decomposition \( n=p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k} \)? To choose any factor \( p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_k^{\alpha_k} \) of \( n \) is the same as to choose the sequence \((\alpha_1, \alpha_2, \ldots, \alpha_k)\) of exponents such that \( \sum_{i=1}^{k} \alpha_i = \alpha \). Then, \( d(n) = |A_1 \times A_2 \times \cdots \times A_k| = |A_1| \cdot |A_2| \cdots |A_k| = (\alpha_1 + 1)(\alpha_2 + 1)\cdots(\alpha_k + 1) \).

### 1.3. Maps

Let \( \mathcal{F}(M, N) \) or \( N^M \) be the set of the mappings \( f \) of \( M \) into \( N \): to each \( x \in M \), \( f \) associates a \( y \in N \), the *image* of \( x \) by \( f \), denoted by \( y=f(x) \). We write often \( f:M \rightarrow N \) instead of \( f \in \mathcal{F}(M, N) \). As \( M \) and \( N \) are finite, \( m=|M|, n=|N| \), we can number the elements of \( M \), so let \( M=\{x_1, x_2, \ldots, x_m\} \). It is clear that giving \( f \) is equivalent to giving a list of \( m \) elements of \( N \), say \((y_1, y_2, \ldots, y_m)\), written in a certain order and with repetitions allowed. By giving the list we mean then that \( y_i \) is the image of \( x_i \), \( 1 \leq i \leq m; y_i=f(x_i) \). In other words, giving \( f \) is equivalent to giving an \( m \)-tuple \( \in N^m \), also called an \( m \)-selection. In this way we find the justification for the notation \( N^M \) for \( \mathcal{F}(M, N) \). Taking \([2a]\) into account, we also have proved the following.

**Theorem A.** The number of maps of \( M \) into \( N \) is given by

\[ [3a] \quad |\mathcal{F}(M, N)| = |N|^M = |N|^{|M|}. \]

For each subset \( A \subseteq M \), we denote:

\[ [3b] \quad f(A) := \{f(x) \mid x \in A\}. \]

In this way a map is defined from \( \mathcal{P}(M) \) into \( \mathcal{P}(N) \), which is called the *extension* of \( f \) to the set of subsets of \( M \). This is also denoted by \( f \).

For all \( y \in N \), the subset of \( M \):

\[ [3c] \quad f^{-1}(y) := \{x \mid f(x) = y\}, \]

which may be empty, is called the *pre-image* or *inverse image* of \( y \) by \( f \).

**Theorem B.** The number of subsets of \( M \), the empty set included, is given by:

\[ [3d] \quad |\mathcal{P}(M)| = 2^{|M|}. \]

Let \( N \) be the set with two elements 0 and 1. We identify a subset \( A \subseteq M \) with the mapping \( f \) from \( M \) into \( N \) defined by: \( f(x) = 1 \) for \( x \in A \), and \( f(x) = 0 \) otherwise (\( f \) is often called the *characteristic function*). In this way we have established a one-to-one correspondence between the sets \( \mathcal{P}(M) \) and \( N^M \); hence, by \([3a]\), \( \mathcal{P}(M) \) has the same number of elements as \( N^M \), which is \(|N|^{|M|} = 2^{|M|} \).

For computing \( u_m = |\mathcal{P}(M)| \), we can also remark that there are just as many subsets of \( M \) that do not contain a given point \( x \) as there are subsets containing it, namely \( u_m - u_{m-1} \) in both cases. Hence \( u_m = u_{m-1} + u_{m-1} = 2u_{m-1} \), which combined with \( u_0 = 1 \) gives \( u_m = 2^m \) indeed. \( \square \)

We recall that \( f \in N^M \) is called *injective* (or is said to be an *injection*) if the images of two different elements are different: \( x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \); \( f \) is called *surjective* (or is said to be a *surjection*) if every element of \( N \) is image of some element in \( M \); \( \forall y \in N, \exists x \in M, f(x) = y \); finally \( f \) is called *bijective* (or is said to be a *bijection*) if \( f \) is surjective as well as injective; in the last case the inverse or reciprocal of \( f \), denoted by \( f^{-1} \), is defined by \( y = f^{-1}(x) \), if and only if \( x = f(y) \), where \( x \in M, y \in N \).

To count a certain finite set \( E \), in other words, to determine the size, consists in principle of constructing a *bijection* of \( E \) onto another set \( F \), whose number of elements is known already; then \( |E| = |F| \).

**Example.** Let \( E \) be the set of all subsets of \( N \) with even size, and \( F \) the set of the others (with odd size). We can choose \( x \in N \) and build a bijection \( f \) of \( E \) into \( F \) as follows: \( f(A) = A \cup \{x\} \) or \( A \setminus \{x\} \) according to \( x \not\in A \) or \( x \in A \). Thus, \( |E| = |F| = \left(\frac{1}{2}\right) |\mathcal{P}(N)| = 2^{n-1} \). (See also p. 13.)

### 1.4. Arrangements, permutations

From now we denote for each integer \( k \geq 1 \):

\[ [4a] \quad \{k\} := \{1, 2, \ldots, k\} = \text{the set of the first } k \text{ integers } \geq 1. \]
DEFINITION A. A k-arrangement \( \alpha \) of a set \( N \), \( 1 \leq k \leq n = |N| \), is an injective map \( \alpha \) from \([k]\) into \( N \) (formerly called 'variation'). We will denote the set of k-arrangements of \( N \) by \( \mathfrak{A}_k(N) \).

Giving such an \( \alpha \) is hence equivalent to giving first a subset of \( k \) elements of \( N \):
\[
B = \{ \alpha(1), \alpha(2), \ldots, \alpha(k) \},
\]
and secondly a numbering from 1 to \( k \) of the elements of \( B \), so finally, a totally ordered subset of \( k \) elements of \( N \), which will often be called a k-arrangement of \( N \) too (not quite correct, but quite convenient).

We introduce now the following notations:

\[
\begin{align*}
[4b] \quad n! &= \prod_{i=1}^{n} i = 1.2.3 \ldots n, \quad \text{if } n \geq 1; \quad 0! := 1. \\
[4c] \quad (n)_k &= \prod_{i=1}^{k} (n - i + 1) = \frac{n!}{(n-k)!} \\
&= n(n-1)\ldots(n-k+1), \quad \text{if } k \geq 1; \quad (n)_0 := 1. \\
[4d] \quad \langle n \rangle_k &= \prod_{i=1}^{k} (n + i - 1) = \frac{(n+k-1)!}{(n-1)!} \\
&= n(n+1)\ldots(n+k-1), \quad \text{if } k \geq 1; \quad \langle n \rangle_0 := 1.
\end{align*}
\]

\( n! \) is called \( n \) factorial; \( (n)_k \) is sometimes called falling factorial \( n \) (of order \( k \)), and \( \langle n \rangle_k \) is sometimes called rising factorial \( n \) (of order \( k \)), or also the Pochhammer symbol. So, \( (n)_k = \langle 1 \rangle_n = n! \), \( \langle n \rangle_k = (n+k-1)_k \), \( (n)_0 = \langle n \rangle_0 = 1 \), etc. These notations are not yet fixed well. The use of \( (n)_k \) in the sense indicated, is inspired by formula \([5a]\) (p. 8) that associates the symbols \( (n)_k \) and \( \binom{n}{k} \) with each other in a symmetrical way, both using parentheses. The symbol \( \langle n \rangle_k \) that we introduce here for lack of any better is not standard, and if often written \( (n)_k \) in texts on hypergeometric series. For the reader familiar with the \( \Gamma \) function:

\[
[4e] \quad n! = \Gamma(n+1), \quad (n)_k = \Gamma(n+1)/\Gamma(n-k+1), \quad \langle n \rangle_k = \Gamma(n+k)/\Gamma(n).
\]

Besides, for complex \( z \) (and \( k \) integer \( \geq 0 \)), \( (z)_k \) and \( \langle z \rangle_k \) still make sense:

\[
[4f] \quad (z)_k := z(z-1)\ldots(z-k+1), \quad (z)_0 := 1 \\
[4g] \quad \langle z \rangle_k := z(z+1)\ldots(z-k+1), \quad \langle z \rangle_0 := 1.
\]

and hence they can be considered as polynomials of degree \( k \) in the indeterminate \( z \).

**Theorem A.** The number of k-arrangements of \( N \), \( 1 \leq k \leq n = |N| \), equals:

\[
[4h] \quad |\mathfrak{A}_k(N)| = (n)_k = n(n-1)\ldots(n-k+1).
\]

- There are evidently \( n \) choices possible for the image \( \alpha(1) \) of 1 (\( \in [k] \)); after the choice of \( \alpha(1) \) is made, there are left only \( (n-1) \) possibilities for \( \alpha(2) \), because \( \alpha \) is injective, so \( \alpha(2) \neq \alpha(1) \); similarly, there are left for \( \alpha(3) \) only \( (n-2) \) possible choices, because \( \alpha(3) \neq \alpha(2) \) and \( \alpha(3) \neq \alpha(1) \), etc.; finally, for \( \alpha(k) \) there are just \( (n-k+1) \) possible choices left. The number of \( \alpha \) is hence equal to the product of all these numbers of choices. This is equal to \( n(n-1)(n-2)\ldots(n-k+1) \).

Note. If \( k > n \), then \( (n)_k = 0 \), and \([4h]\) is still valid.

**Definition B.** A permutation of a set \( N \) is a bijective map of \( N \) onto itself. We denote the set of permutations of \( N \) by \( \mathfrak{S}(N) \).

**Theorem B.** The number of permutations of \( N \), \( |N| = n \geq 1 \), equals \( n! \).

- One can argue as in the proof of Theorem A above. One may also observe that there is a bijection between \( \mathfrak{S}(N) \) and \( \mathfrak{A}_n(N) \).

1.5. Combinations (without repetitions) or Blocks

**Definition A.** A k-combination \( B \), or k-block, of a finite set \( N \) is a nonempty subset of \( k \) elements of \( N \): \( B \subset N \), \( 1 \leq |B| \leq n = |N| \). If one does not know in advance whether \( k \geq 1 \), one says rather k-subset of \( N \) (\( k \geq 0 \)). We denote the set of k-subsets of \( N \) by \( \mathfrak{B}_k(N) \).

A k-block is also called a combination of \( k \) to \( k \) of the \( n \) elements of \( N \). Pair and 2-block are synonymous; similarly, triple or triad and 3-block, etc.

Next we show three other ways to specify a k-subset of \( N \), \( |N| = n \).

**Theorem A.** There exists a bijection between \( \mathfrak{B}_k(N) \) and the set of functions \( \varphi : N \rightarrow \{0,1\} \), for which the sum of the values equals \( k \).

**Theorem B.** There exists a bijection between \( \mathfrak{B}_k(N) \) and the set of solu-
Theorem C. Giving a \( B \in \mathfrak{P}_k(N) \) is equivalent to giving a distribution of \( k \) indistinguishable balls in \( n \) distinct boxes, each box containing at most one ball.

For Theorem A it is sufficient to define for each \( B \in \mathfrak{P}_k(N) \) the characteristic function \( q = q_B \) by \( q(y) = 1 \) if \( y \in B \) and \( = 0 \) otherwise. For Theorem B we number the elements of \( N \) from 1 to \( n \), \( N = \{ y_1, y_2, \ldots, y_n \} \); for each \( B \in \mathfrak{P}_k(N) \) we define \( x_i = x_i(B) \) by \( x_i = 1 \) if \( y_i \in B \) and \( = 0 \) otherwise. Finally, for Theorem C each box is associated with a point \( y \in N \); to every \( B \in \mathfrak{P}_k(N) \) we associate the following distribution: the box associated with \( y \) contains a ball if \( y \in B \) and no ball if \( y \notin B \).

Theorem D. The number of \( k \)-subsets of \( N \), \( 0 \leq k \leq n = |N| \), denoted by \( \binom{n}{k} \), equals:

\[
\binom{n}{k} = \frac{|\mathfrak{P}_k(N)|}{k!} = \frac{n(n-1) \ldots (n-k+1)}{k!} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}.
\]

We will adopt the notation \( \binom{n}{k} \), used almost in this form by Euler, and fixed by Raabe, with the exclusion of all other notations, as this notation is used in the great majority of the present literature, and its use is even so still increasing. This symbol has all the qualities of a good notation: economical (no new letters introduced), expressive (it is very close in appearance to the explicit value \( \frac{n!}{k!(n-k)!} \)), typical (no risk of being confused with others), and beautiful. In certain cases, one might prefer \( (a, b) \) instead of \( \binom{a+b}{a} \) (see pp. 27 and 28), so that \( (a, b) = (b, a) \) is perfectly symmetric in \( a \) and \( b \). We recall anyway the ‘French’ notation \( \binom{a}{b} \), and the ‘English’ notation \( ^aC_b \).

We prove equality (\( * \)); the others are immediate consequences. If \( k = 0 \), \( (n)_0/0! = 1 \) \( [4b, c] \) (p. 6), and \( |\mathfrak{P}_k(N)| = 1 \) because \( \mathfrak{P}_k(N) \) contains only the empty subset of \( N \). Let us suppose \( k \geq 1 \). With every arrangement \( \sigma \in \mathfrak{P}_k(N) \), we associate \( B = f(\sigma) = \{ a(1), a(2), \ldots, a(k) \} \in \mathfrak{P}_k(N) \) (p. 6). \( f \) is a map from \( \mathfrak{P}_k(N) \) into \( \mathfrak{P}_k(N) \) such that for all \( B \in \mathfrak{P}_k(N) \) we have:

\[
|f^{-1}(B)| = k!,
\]

since there are \( k! \) possible numberings of \( B \) (the number of \( k \)-arrangements of \( B \)). Now the set of pre-images \( f^{-1}(B) \), which are mutually disjoint, covers \( \mathfrak{P}_k(N) \) entirely as \( B \) runs through \( \mathfrak{P}_k(N) \). Hence, the number of elements of \( \mathfrak{P}_k(N) \) equals the sum of all \( |f^{-1}(B)| \), where \( B \in \mathfrak{P}_k(N) \), which is \( \sum (\binom{n}{k}) \) (\( * \)). Hence, by \([4h] \) (p. 7) for equality (\( ** \)), and by \([5b] \) for (\( *** \)):

\[
\binom{n}{k} = \frac{|\mathfrak{P}_k(N)|}{k!} = \sum |f^{-1}(B)| = k! \cdot |\mathfrak{P}_k(N)|.
\]

The argument we just have used is sometimes called the ‘shepherd’s principle’: for counting the number of sheep in a flock, just count the legs and divide by 4.

Definition B. The integers \( \binom{n}{k} \) are called binomial coefficients.

We will see the justification of this name on p. 12.

Definition C. The double sequence \( \binom{n}{k} \) which is defined by \([5a] \) for \( (n, k) \in N^2 \) (and equal to 0 for \( k > n \)) will be defined from now on also for \((x, y) \in C^2 \) in the following way:

\[
\binom{x}{y} = \begin{cases} (x)_y & \text{if } x \in C, \ y \in N \\ y! & \text{if } x \in C, \ y \notin N \end{cases}
\]

where \( (x)_k := x(x-1) \ldots (x-k+1) \) for any \( k \in N \). \((x)_0 = 1 \).

We will constantly use this convention in the sequel.
Theorem E. The binomial coefficients satisfy the following recurrence relations:

\[ \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} ; \quad k, n \geq 1. \]  

\[ \binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1} + \frac{n-k+1}{k} \binom{n}{k-1} ; \quad k, n \geq 1. \]  

\[ \binom{n+1}{k+1} = \binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \sum_{i=0}^{n-k} \binom{i}{k} ; \quad k, n \geq 0. \]  

If \( n \) is replaced by a real or complex number \( z \) (\( z \) can also play the role of an indeterminate variable), then \([5e, f, h]\) still hold, and we have instead of \([5g]\), for each integer \( s \geq 0 \):

\[ \binom{z+1}{k+1} = \binom{z}{k} + \binom{z-1}{k} + \cdots + \binom{z-s}{k} + \binom{z-s}{k+1}. \]

\( [5e, f] \) can be verified by substituting the values \([5a, d]\) for the binomial coefficients; \([5g']\), hence \([5g]\), follows by applying \([5e]\) to each of the terms of the sum \( \sum_{i=0}^{s} \binom{z+1}{k+1} \) followed by the evident simplification. For \([5h]\), an analogous method works (a generalization is found at the end of Exercise 30, p. 169).

As an example, we will also give combinatorial proofs of \([5e, f, g]\).

For \([5e]\), let us choose a point \( x \in N, |N| = n \), and let \( \mathcal{S} \) and \( \mathcal{F} \) respectively be the system of \( k \)-blocks of \( N \) that contain or not contain respectively the point \( x \). Clearly, \( \mathcal{S} \cap \mathcal{F} = \emptyset \), so:

\[ \mathcal{B}_k(N)| = |\mathcal{S}| + |\mathcal{F}| \]

Now every \( B \in \mathcal{S} \) corresponds to exactly one \( B' \in \mathcal{B}(N \setminus \{x\}) \), namely \( B \setminus \{x\} \), hence:

\[ |\mathcal{S}| = |\mathcal{B}_{k-1}(N \setminus \{x\})| = \binom{n-1}{k-1}. \]

Also, \( \mathcal{F} = \mathcal{B}_k(N \setminus \{x\}) \); hence:

\[ \binom{n}{k} = \binom{n-1}{k}. \]

Finally, \([5i, j, k]\) imply \([5e]\).

For \([5f]\), let us take the interpretation of \( \binom{n}{k} \) as the number of distributions of balls in boxes (Theorem C, p. 8). We form all the \( \binom{n}{k} \) distributions successively. Then we need in total \( k \binom{n}{k} \) balls. The \( n \) boxes play a symmetric role, so every box receives \( \frac{1}{n} k \binom{n}{k} \) times a ball.

Now, every distribution that gives a ball to a given box, corresponds to exactly one distribution of \( (k-1) \) balls in the remaining \( (n-1) \) boxes. These are \( \binom{n-1}{k-1} \) in number, so as result we find that \( k/n \binom{n}{k} = \binom{n-1}{k-1} \).

For \([5g]\), we number the elements of \( N, N':= \{x_1, x_2, \ldots, x_n\} \). We put for \( i=1, 2, \ldots \):

\[ \mathcal{S}_i := \{ B \in \mathcal{B}_k(N); x_1, x_2, \ldots, x_{i-1} \notin B; x_i \in B \}. \]

Evidently, each \( B \in \mathcal{B}_k(N) \) belongs to exactly one \( \mathcal{S}_i \), \( i \in [n] \). So:

\[ \binom{n}{k} = |\mathcal{S}_1| + |\mathcal{S}_2| + \cdots. \]

Now, every \( B \in \mathcal{S}_i \) corresponds to exactly one:

\[ B' := B \setminus \{x_i\} \in \mathcal{B}_{k-1}(N \setminus \{x_1, x_2, \ldots, x_i\}). \]

Hence:

\[ \binom{n-1}{k-1} = |\mathcal{B}_{k-1}(N \setminus \{x_1, x_2, \ldots, x_i\})| = \binom{n-1}{k-1}. \]

and we see that \([5m]\) imply \([5g]\).

Pascal triangle (or arithmetical triangle) is the name for the infinite table, which is obtained by placing each number \( \binom{n}{k} \) at the intersection of the \( n \)-th row and the \( k \)-th column, \( k, n \geq 0 \) (Figure 1). The numerical computation of the first values can be quickly done, by using \([5e]\) and the initial values \( \binom{0}{0} = 1 \), except for \( \binom{0}{k} = 0 \).

Each recurrence relation \([5e, f, g]\) can be advantageously visualized by a diagram (Figures 2a, b, c): in every Pascal triangle represented by the
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shaded area, the heavy dots represent the pairs \((n, k)\) such that the corresponding \({n \choose k}\) are related by a linear recurrence relation (that is, with coefficients that are possibly not constant with respect to \(n\) and \(k\), as, for example \([5f]\)). Diagrams 2a, b, c are said to be of the second, first and \((n-k)\)-th order, respectively, as their associated recurrence relations. A table of binomial coefficients is presented on p. 306.

1.6. BINOMIAL IDENTITY

THEOREM A. (Newton binomial formula, or binomial identity). If \(x\) and \(y\) are commuting elements (\(xy=yx\)) of a ring, then we have for each integer \(n \geq 0\):

\[
\begin{align*}
(x + y)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \\
&= x^n + \left( \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots \right)
\end{align*}
\]

Note. If the ring does not have an identity, we must interpret \(x^0 y^n\) and \(x^n y^0\) as \(y^n\) and \(x^n\), respectively. We can also consider \([6a]\) as an identity between polynomials of the indeterminates \(x\) and \(y\).

Let us examine the coefficients \(c_{k,1}\) of the expansion of:

\[
\begin{align*}
\binom{n}{k} &= P_1 P_2 \ldots P_n = \sum_{k=1}^{n} c_{k,1} x^{k} y^{n-k}, \\
P_i &= x + y, \quad i \in [n].
\end{align*}
\]

The term \(x^k y^{n-k}\) is obtained by choosing \(k\) of the \(n\) factors \(P_i, i \in [n]\), in the sense that one multiplies the terms \('x'\) of these factors by the terms \('y'\) of the remaining \((n-k)\) factors. So \(l = n - k\). Hence the coefficient \(c_{k,1} = c_{k,n-k}\) equals the number of different choices of the \(k\) factors \(P_i\) among the \(n\), hence equals \(\binom{n}{k}\) (Theorem D, p. 8).

For instance, if \(x = y = 1\), then we have \(\sum \binom{n}{k} = 2^n\) and thus we find again the result of p. 5: the total number of subsets of \(N\) equals \(2^n\).

If \(x = -1, y = 1\) we obtain \(\sum (-1)^k \binom{n}{k} = 0\), in other words: in \(N\) there are just as many 'even' as 'odd' subsets (see also p. 5).

Now we evaluate the \(n\)-th power of the difference operator.

THEOREM B. Let \(A\) be the difference operator, which assigns to every function \(f \in A^R\), defined on the real numbers, and with values in a ring \(A\), the function \(g = Af\), which is defined by \(g(x) = f(x+1) - f(x), x \in R\). For each integer \(n \geq 2\), we define \(A^n f = Af (Af)^{n-1} f\), and we denote \(A^n f(x)\) instead of \((Af)^n(x)\). Then we have:

\[
A^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+k), \quad n = 0, 1, 2, \ldots
\]

Let \(E\) be the translation operator defined by \(Ef(x) = f(x+1)\), and \(I\) the identity operator, \(I = f\). Clearly, \(A = E - I\). Now \(E\) and \(I\) commute in the ring of operators acting on \(A^R\). Hence, defining \(E^{k} = E(E^{k-1}) = E(E^{k-2}) = \ldots\), we have, by \([6a]\):

\[
A^n = (E - I)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} E^k
\]

(since \(I^{n-k} = I\)), from which \([6c]\) follows, as \(E^k f(x) = f(x+k)\).
In the case of a sequence \( u_m, m \in \mathbb{N} \), \([6c]\) implies:

\[
A^n u_m = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} u_{m+k}^k,
\]

where \( A u_m = u_{m+1} - u_m, \) \( A^2 u_m = A(A u_m) = u_{m+2} - 2u_{m+1} + u_m, \) etc. \( A u_m \geq 0, \)
for all \( m \), means that \( u_m \) is increasing, \( A^2 u_m \geq 0 \) for all \( m \), means that \( u_m \) is convex.

If \( A \) operates on one of the variables of a function of several variables, one can place a dot over the variable concerned to indicate this. So we write:

\[
[6e] \quad A f(u, v) := f(u + 1, v) - f(u, v),
\]

\[
A f(u, v) := f(u, v + 1) - f(u, v).
\]

Examples. \((1) \ A^0 u^m = \) the value of \( A^m X^m \) in the point \( x=0 \), and then \([6c]\) gives:

\[
[6f] \quad A^0 u^m = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (k - j)^m,
\]

which are, up to a coefficient \( k! \), the Stirling numbers of the second kind (cf. p. 204).

\[
(2) \quad A^n = \frac{1}{(x)_n} (x + n)^k \quad \text{(by induction)}.
\]

We cite also the following interesting arithmetical property of binomial coefficients:

**Theorem C.** For each prime number \( p \), we have:

\[
[6g] \quad \binom{p}{k} \equiv 0 \pmod{p}, \quad \text{except} \quad \binom{p}{0} = \binom{p}{p} = 1.
\]

In other words:

\[
[6g'] \quad (1 + x)^p \equiv 1 + x^p \pmod{p},
\]

which means that these two polynomials have the same coefficients in \( \mathbb{Z}/p\mathbb{Z} \).

(Exercise 17, p. 78 gives many other arithmetical properties of the binomial coefficients.)

\[
\text{As} \quad \binom{p}{k} = \frac{p!}{k!(p-k)!} \quad \text{is an integer,} \quad k! \text{ divides } (p-1)(p-2) \cdots (p-k+1), \text{ and it is}
\]

relatively prime with respect to \( p \) if \( 1 \leq k \leq p-1 \); hence it divides \( (p-1)(p-k) \) according to the theorem of Gauss. Thus, \( (p-1)(p-k) \equiv 0 \pmod{p} \).

\[1.7. \quad \text{Combinations with repetitions}
\]

**Definition.** A \( p \)-combination with repetition \( T \), or unordered \( p \)-selection, or \( p \)-CR of a finite set \( N \), is a list of \( p \) elements, all taken from \( N \), repetitions allowed, but not the order in the list taken into account. We denote the set of \( p \)-CR of \( N \) by \( \Omega_p(N) \).

For example, \( \{a, b, a, b, c\} \) and \( \{b, b, a, a\} \) are identical \( 5 \)-CR of \( \{a, b, c\} \). Each \( k \)-block of \( N \) can be considered as a \( k \)-CR of \( N \).

**Theorem A.** There exists a bijection between \( \Omega_p(N) \) and the set of functions \( \psi : \mathbb{N} \rightarrow \mathbb{N} \) for which the sum of the values \( \psi \) equals \( \sum_{x \in \mathbb{N}} \psi(x) \).

**Theorem B.** There is a bijection between \( \Omega_p(N) \) and the set of integer solutions, consisting of integers \( \geq 0 \), of the equation:

\[
[7a] \quad x_1 + x_2 + \cdots + x_p = p.
\]

Each solution of \([7a]\) is also called 'composition of \( p \) into \( n \) summands' (see Exercise 23, p. 123).

**Theorem C.** To each \( T \in \Omega_p(N) \) corresponds exactly one distribution of \( p \) indistinguishable balls into \( n \) distinct boxes.

The reasoning is the same as for Theorems A, B, and C, on p. 7.

\[\text{For Theorem A, define for each } T \in \Omega_p(N) \text{ the function } \psi = \psi_T : \mathbb{N} \rightarrow \mathbb{N} \text{ by } \psi(y) = \text{the number of times that } y \text{ appears in } T. \text{ For Theorem B, } \]

\[N = \{v_1, v_2, \ldots, v_n\} \text{ and } x_i = \psi_T(v_i). \text{ For Theorem C, identify each point } x \in N \text{ with a box.}\]

**Theorem D.** The number of \( p \)-CR of a finite set \( N, |N| = n \geq 1, p \geq 0 \) equals
We give two proofs of this theorem.

(1) We partition the set of solutions of [7a] (we denote the number of these solutions by $T(n, p)$) into two kinds. First, the solutions with $x_1 = 0$; there are evidently $T(n-1, p)$ of them. Next, the solutions for which $x_1 \geq 1$; if for these we put $x'_1 = x_1 - 1 \geq 0$, these solutions correspond each to exactly one solution of $x'_1 + x_2 + \cdots + x_n = p - 1$, of which there are $T(n, p - 1)$. Finally,

$$T(n, p) = T(n - 1, p) + T(n, p - 1).$$

To this relation we still must add the following initial conditions, which follow from [7a]:

$$T(n, 0) = 1, \quad T(1, p) = 1.$$

Now the double sequence $T(n, p)$ is completely determined. As a matter of fact, the sequence $W(n, p) = \binom{n + p - 1}{p}$ evidently satisfies the recurrence relation [7c] as well as the 'boundary condition' [7d]. Hence $T(n, p) = W(n, p)$.

(1I) We represent the $n$ boxes $x_1, x_2, \ldots, x_n$ of Theorem C in a row, side by side. We number the separations between the boxes by $c_1, c_2, \ldots, c_{n-1}$, going from left to right (Figure 3). Let now $N = \{x_1, x_2, \ldots, x_n\}$ be the set of these boxes and let $Z = \{n + p - 1\} - \{1, 2, \ldots, n + p - 1\}$. Now we define the map $f$ from $\Omega_p(N)$ into $\Omega_{n-1}(Z)$ as follows: with every distribution of balls associated with $T \in \Omega_p(N)$, we associate the $(n - 1)$-block

\[
\begin{bmatrix}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{bmatrix}
\]

Fig. 3. $n = 5$, $p = 9$.

For any variable $x$, \begin{align*}
\binom{x}{k} &= \binom{x + k - 1}{k} = \frac{x(x + 1) \cdots (x + k - 1)}{k!}.
\end{align*}

Then we have properties for $\binom{x}{k}$ very analogous to those of Theorem E on p. 10. For example:

\[
\begin{align*}
\binom{x}{k} &= \binom{x}{k - 1} + \binom{x}{k} \\
\binom{x + y}{k} &= \binom{x}{k - 1} + \binom{y}{k} \\
\binom{x + 1}{k} &= \sum_{j=0}^{x-1} \binom{j}{k}
\end{align*}
\]
Abelian words. One can also give a more abstract definition of the concept of combination with repetitions, which is important to know. Let \( \mathcal{X} \) be a nonempty set, the alphabet: we denote the set of finite sequences \( f \) of elements of \( \mathcal{X} \) (also called letters) by \( \mathcal{X}^* \). We denote \( f = (x_1, x_2, \ldots, x_r) \), where \( r \) is a variable integer \( \geq 1 \). Such a sequence \( f \) is also called an \( r \)-arrangement with repetition of \( \mathcal{X} \). Hence, when \( \mathcal{X} = \{1\} \) has the meaning given on p. 4, and when we make the convention to let the empty set 0, denoted by 1, also belong to \( \mathcal{X}^* \), then we have:

\[
\mathcal{X}^* = \{1\} \cup \bigcup_{r \geq 1} \mathcal{X}^r.
\]

The sequence \( f = (x_1, x_2, \ldots, x_r) \) will be identified with the monomial or word \( x_1 x_2 \ldots x_r \). In this form, the integer \( r \) is called the degree of the monomial or the length of the word \( f \). By definition, the length of 0 is 0.

In the case \( \mathcal{X} \) is finite, \( \mathcal{X} = \{x_1, x_2, \ldots, x_n\} \), we can denote by \( a_i \) the number of times that the letter \( x_i \) occurs in the word \( f \), \( a_i \geq 0 \), \( i \in [n] \); in that case we often say that \( f \) has the specification \( (a_1, a_2, \ldots, a_n) \). For example, for \( \mathcal{X} = \{x, y\} \), \( \mathcal{X}^{[3]} \) consists of the following 8 words: \( xxx, xyx, yxx, xyy, yyx, yxy, yyx \). These can also be written: \( x^3, x^2y, xyx, y^2x, xy^2, yxy, yyx, yyx \). The specifications are then \((3,0)\), \((2,1)\), \((2,1)\), \((2,1)\), \((1,2)\), \((1,2)\), \((1,2)\), \((0,3)\), respectively.

The set \( \mathcal{X}^* \) is equipped with an associative composition law, the product by juxtaposition, which associates with two words \( f = x_1 x_2 \ldots x_r \) and \( g = y_1 y_2 \ldots y_s \), the product word \( fg = x_1 x_2 \ldots x_r y_1 y_2 \ldots y_s \), where \( x_n = y_n \) if \( t \leq r \), and \( x_n = x_1 \) if \( t > r \). One also says that \( fg \) is the concatenation of \( f \) and \( g \). This composition law is associative, and has the empty word 1 as unit element. In this way \( \mathcal{X}^* \) becomes a monoid (that is to say a set with an associative multiplication, and a unit element), which is called the free monoid generated by \( \mathcal{X} \). Furthermore, when we denote the set of words of length \( n \) by \( \mathcal{X}^{[n]} \), we identify \( \mathcal{X}^{[n]} \) with \( \mathcal{X} \).

We introduce an equivalence relation on \( \mathcal{X}^* \), by defining two words \( f \) and \( g \) to be equivalent if and only if they consist of the same letters, up to order, but with the same number of repetitions. The equivalence class that contains \( f \), is called the abelian class of \( f \), or also the abelian word \( f \). There is a one-to-one correspondence between the abelian classes and the maps \( \psi \) from \( \mathcal{X} \) into \( \mathbb{N} \) that are everywhere zero except for a finite number of points. In fact, if we index the set \( E \) of the \( y \in \mathcal{X} \) where \( \psi(y) > 0 \), in such a way that \( E = \{y_1, y_2, \ldots, y_l\} \), then we can bijectively associate with \( \psi \) the abelian class of the word:

\[
y_1^{\psi(y_1)} y_2^{\psi(y_2)} \cdots y_l^{\psi(y_l)} = \psi(y_1) \psi(y_2) \cdots \psi(y_l) \times \psi(y_1) \times \psi(y_2) \times \psi(y_l).
\]

If \( \mathcal{X} \) is finite, \( \mathcal{X} = \mathbb{N} \), it is clear that an abelian word is just a combination with repetitions, of \( \mathbb{N} \) (Definition, p. 15).

The set of abelian words \( \mathcal{X}^* \) can also be made into a monoid, when we consider it as a part of \( \mathbb{N}^* \); this last set is equipped with the usual addition of functions \( \psi \). In this way we define the free abelian monoid generated by \( \mathcal{X} \).

1.8. SUBSETS OF \([n]\), RANDOM WALK

Let \( N \) be a finite totally ordered set (Definition D, p. 59), with \( n \) elements, which we identify with \( [n] := \{1, 2, \ldots, n\} \). We are going to give several interpretations to the specification of a subset \( p \subset [n] \), of cardinal \( p(-[P]) \). We introduce moreover:

\[
g := |P| - |P| = n - p.
\]

(1) To give a \( P \subset [n] \) is equivalent to giving an integer-valued sequence \( x(t) \), defined by:

\[
x(t) - x(t - 1) = \begin{cases} 1 & \text{if } t \in P; \ t \in [n], \\ -1 & \text{if } t \notin P; \ t \in [n], \\ 0 & \text{otherwise.} \end{cases}
\]

One can represent \( x(t) \) by a broken line, which is straight between the points with coordinates \( (t, x(t)) \). Thus, Figure 4 represents the \( x(t) \) associated with the block:

\[
P = \{3, 5, 6, 7, 8, 10, 11, 12\} \subset [12].
\]
Evidently, \( p + q = n \) and \( x(n) = (x(n) - x(n-1)) + \cdots + (x(2) - x(1)) + x(1) = p - q \); hence:

\[
[8c] \quad p = \frac{1}{2} (n + x(n)), \quad q = \frac{1}{2} (n - x(n)).
\]

This way of determining \( P \subseteq [n] \) suggests a process, if we imagine that \( t \) represents successive instants 1, 2, ..., \( n \).

(2) Giving \( P \subseteq [n] \) is also equivalent to giving the results of a game of heads or tails, played with \( n \) throws of a coin, if we agree that

\[
x(t) - x(t-1) = 1 \iff \text{the } t\text{-th throw is tails} \quad (t \in [n]).
\]

The numbers \( p, q \) of \([8c]\) are then the numbers of tails and heads obtained in the course of the game, respectively. Because of this interpretation, the sequence \( x(t) \) is often called *random walk*: it translates the (stochastic) movements by jumps of \( \pm 1 \) of a moving point on the \( x \)-axis, whose motion occurs only at the times \( t = 1, 2, \ldots, n \) (a kind of Brownian movement on a line).

Giving \( P \subseteq [n] \) is also equivalent to giving the successive results of drawing balls from a vase, which contains \( p \) black and \( q \) white balls, and agreeing that \( x(t) - x(t-1) = 1 \) if the \( t\)-th ball drawn is black \( (t \in [n]) \).

(3) One often prefers in combinatorial analysis to represent \( P \subseteq [n] \) by a polygonal line \( \varphi \) which joins the origin \((0,0)\) with the point \( B \) with coordinates \((p, q)\) such that the horizontal sides, having lengths one and also called *horizontal steps*, correspond to the points of \( p \), and the vertical sides correspond to the points of the complement of \( p \). Thus, Figure 5 represents the subset \( p \) defined by \([8b]\). Such a polygonal line may be called *minimal path* joining \( O \) to \( B \) (of length \( n = p + q \)). (In fact, there does not exist a shorter path of length less than \( n \), which joins \( O \) to \( B \), consisting of unit length straight sections bounded by points with integer coordinates.)

(4) Finally, giving \( P \subseteq [n] \) is also equivalent to giving a *word* \( f \) with two letters \( a \) and \( b \), of length \( n \), where the letter \( a \) occurs \( p \) times, and the letter \( b \) occurs \( q \) times, \( p = |P| \) (see p. 18). Thus, the word representing \( P \) of \([8b]\) is \( bbababaaaaa \).

Now we treat two examples of enumerations in \([n]\).

**Theorem A.** ([Gergonne, 1812], [Muir, 1901]). Let \( f_i(n, p) \) be the number of \( p \)-blocks \( P \subseteq [n] \) with the following property: between two arbitrary points of \( P \) are at least \( l \) \( \geq 0 \) points of \([n]\) which do not belong to \( P \). Then:

\[
[8d] \quad f_i(n, p) = \binom{n - (p - 1)!}{p}.
\]

Let \( P \) be \( \{i_1, i_2, \ldots, i_p\} \), \( 1 \leq i_1 < i_2 < \cdots < i_p \leq n \) and \( y_k := i_k - i_{k-1} - 1 \), \( y_1 := i_1 - 1 \), \( y_{p+1} := n - i_p \). Giving \( P \) is equivalent to giving a solution with integers \( y_i \) of:

\[
[8e] \quad y_1 + y_2 + \cdots + y_p + y_{p+1} = n - p \quad y_k \geq l \quad \text{if} \quad 2 \leq k \leq p, \quad y_1 \quad \text{and} \quad y_{p+1} \geq 0.
\]

We put \( z_1 := y_{l-1} \) if \( 2 \leq k \leq p \), and \( z_1 := y_1, \quad z_{p+1} := y_{p+1} \). Then \( z_i \geq 0 \), for every \( i \in [p + 1] \) and \([8e]\) is equivalent to:

\[
[8f] \quad z_1 + z_2 + \cdots + z_p + z_{p+1} = n - p - (p - 1)! \quad \text{which has} \quad \binom{n - (p - 1)!}{p} \quad \text{solutions}, \quad \text{by Theorems B and D, of pp. 15.}
\]

Observe that \( l = -1 \) recovers \([7b]\) p. 16...

(For other problems concerning the blocks of \([n]\), the reader is referred to [*David, Barton, 1962]*, pp. 85–101, [Abramson, 1964, 1965], [Abramson, Moser, 1960, 1969], [Church, Gould, 1967], [Kaplansky, 1943, 1945], [(René) Lagrange, 1963], [Mood, 1940].)

**Theorem B (of André).** Let \( p \) and \( q \) be integers, such that \( 1 \leq p < q, p + q = n \).

The number of minimal paths joining \( O \) with the point \( M(p, q) \) (in the sense of (3) on p. 20) that do not have any point in common with the line \( x = y \), except the point \( O \), is \( \frac{q - p}{q + p} \binom{n}{p} \). In other words, if there is a ballot, for
which candidates \( \mathcal{P} \) and \( \mathcal{Z} \) receive \( p \) and \( q \) votes respectively (so \( \mathcal{Z} \) is elected), then the probability that candidate \( \mathcal{Z} \) has constantly the majority during the counting of the votes is equal to \( \frac{(q - p)}{(q + p)} \).

This is the famous ballot problem, formulated by [Bertrand, 1887]; we give the elegant solution of [André, 1887]. Désiré André, born Lyon, 1840, died Paris, 1917, devoted most of his scientific activity to combinatorial analysis. A list and a summary of his principal works are found in [André, 1910]. See also Exercises 11 and 13 pp. 258 and 260.

We first formulate the principle of reflection, which essentially is due to André. Let be given a line \( D \) parallel to the line \( x = y \), and two points \( A, B \) lying on the same side of \( D \) (for instance above, as in Figure 6). The number of minimal paths (the adjective minimal will be omitted in the sequel) joining \( A \) with \( B \) that intersect or touch \( D \), is equal to the number of paths joining \( B \) with the point \( A' \) which lies symmetric to \( A \) with respect to \( D \). In fact, when \( \mathcal{V} \) stands for the first point that \( \mathcal{V} \) has in common with \( D \), going from \( A \) to \( B \), we can let the path \( \mathcal{V} = (A, Z, B) \) correspond to the path \( \mathcal{Q}' = (A', Z, B) \), which is just the same as \( \mathcal{V} \) between \( I \) and \( B \), but with the part \( A'Z \) just equal to the image by reflection with respect to \( D \) of the part \( AZ \) of \( \mathcal{V} \).

Now let \( C(A, B) \) be the set of paths joining \( A (x_A, y_A) \) with \( B (x_B, y_B) \), \( 0 \leq x_A \leq x_B, 0 \leq y_A \leq y_B \). Clearly, the number of paths joining \( A \) with \( B \) equals:

\[
|C(A, B)| = \frac{(x_B - y_B - x_A - y_A)}{(x_B - y_A)},
\]

because giving a path is equivalent to choosing a set of \( (x_B - x_A) \) horizontal segments among \( (x_B + y_B - x_A - y_A) \) places (the duration of the walk).

Let us call a suitable path one that satisfies the hypotheses of Theorem B. The number of suitable paths, which is the number of paths joining \( W(0, 1) \) with \( B(p, q) \) without intersecting the line \( x = y \), is hence, by the principle of reflection equal to \( |C(W, B)| - |C(V, B)| \) (Figure 7); which means, by [8g], equal to \( \left( \frac{p + q - 1}{p} \right) - \left( \frac{p + q - 1}{p-1} \right) \), hence the result, after simplifications.

The probabilistic interpretation supposes that every path \( \varepsilon C(O, B) \) is equally probable, so that the probability we look for is the quotient of the number of suitable paths (which we found already), and the total number of paths joining \( O \) with \( B \), which is \( |C(O, B)| = \left( \frac{n}{p} \right) \): we find that the probability is \( \frac{(q - p)}{(q + p)} \), as announced. Every step represents a vote, the horizontal ones being for \( \mathcal{P} \) and the vertical ones for \( \mathcal{Z} \). ■


### 1.9. Subsets of \( \mathbb{Z}/n\mathbb{Z} \)

Let \( N \) be a finite set of \( n \) points placed on a circle with equal distances between two adjoining points. We identify this set with the set of residue classes modulo \( n \), denoted by \([n]\):
Theorem ([Kaplansky, 1943]). Let $g_l(n, p)$ be the number of $p$-blocks $P \subseteq [n]$ with the following property: between any two points $v$ and $w$ of $P$ (that means on each of the two open arcs $vw$ of the circle on which we think $[n]$ situated) there are at least $l$ points of $[n]$ that do not belong to $P$. Then:

$$g_l(n, p) = \frac{n}{n-2(l-1)} \left(1 - \frac{p}{n-1}\right).$$

When $\mathcal{A}$ stands for the set of the $P \subseteq [n]$ that satisfy the condition mentioned in the theorem, $[I] := \{0, 1, \ldots, I-1\}$, then we let:

$$\mathcal{A}_l := \{p \mid p \in \mathcal{A}, p \cap [I] = \emptyset\}, \quad I = 0, 1, 2, \ldots, I-1.$$

$$\mathcal{A} := \mathcal{A} \cup \mathcal{A}^*, \quad \mathcal{A} := \{p \mid P \in \mathcal{A}, P \cap [I] = \emptyset\}.$$

$\mathcal{A}^*$ and the $\mathcal{A}_l$ evidently partition $\mathcal{A}$ into $I+1$ disjoint subsets. Hence:

$$g_l(n, p) = |\mathcal{A}| = |\mathcal{A}^*| + \sum_{i=0}^{I-1} |\mathcal{A}_i|. $$

Now, choosing $P \in \mathcal{A}_I$ is equivalent to choosing on the straight interval $[I+1, I+2, \ldots, n-I]$ the $p-1$-block $P' := P \setminus \{i\}$ with $n-2l-1$ elements. Hence, by Theorem A (p. 21), we have:

$$|\mathcal{A}| = f_l(n-2l-1, p-1), \quad 0 \leq l \leq I-1.$$

Similarly, choosing $P \in \mathcal{A}^*$ is equivalent to choosing it on the straight interval $[I+1, I+2, \ldots, n-I]$ with $n-l$ elements. Hence:

$$|\mathcal{A}^*| = f_l(n-l, p).$$

Finally, $[9c, d, e]$ imply, by $[8d]$ (p. 21) for the equality $(*), and with simplifications for ($**$):

$$g_l(n, p) = f_l(n-2l-1, p-1) + f_l(n-l, p) - \frac{(n-p^l-1)}{p^l} + \frac{n-2l-1}{p^l} = \frac{n}{n-2(l-1)} \left(1 - \frac{p}{n-1}\right).$$

It would be interesting to give a combinatorial significance of $g_l(n, p)$ for $l<0$. Also see Exercise 40, p. 173.
Many identities are only the consequence of [lob]: one counts a set in two different ways, which gives a combinatorial proof of the identity which is to be examined.

**Examples.** (1) Let \(E\) be the set of nonempty subsets of \(A := \{1, 2, \ldots, m+1\}\), and let us call \(E_j (E)\) the set of subsets of \(A\) for which the greatest element is \(j \in \{1, 2, \ldots, m+1\}\). Evidently, \(E = \bigcup_{j=1}^{m+1} E_j\). Now, \(|E_j| = 2^{m+1} - 1\) and \(|E_j| =_{j=1}^{m+1} 2^{j-1}\) (the number of subsets of \(\{1, 2, \ldots, j-1\}\)). Then, by using [lob] we obtain: \(2^{m+1} - 1 = 1 + 2 + 2^2 + \cdots + 2^m\). More generally, for any integers \(x, y, m \geq 1\), we could prove by a strictly combinatorial argument the well-known identity:

\[
x^{m+1} - y^{m+1} = (x - y)(x^m + x^{m-1}y + \cdots + y^m).
\]

(2) Let \(Z = X + Y\) be a division of the set \(Z\), \(x := |X| \geq 0\), \(y := |Y| \geq 0\). We denote \(E \subseteq Z\) for the set of all \(A \subseteq Z\) such that \(|A| = n\) \((E = \mathbb{P}_n(Z))\), and \(E_k\) for the set of all \(B \subseteq E\) such that \(|B| = k\). Clearly, \(E = \bigcup_{k=0}^{n} E_k\). Now, from \(|E| = \binom{x+y}{n}\) and \(|E_k| = \binom{x}{k} \binom{y}{n-k}\) follows the Vandermonde convolution (see p. 44):

\[
\binom{x+y}{n} = \sum_{k=0}^{n} \binom{x}{k} \binom{y}{n-k}.
\]

(3) With \(Z = X + Y\) once again, let \(E\) be the set of functions \(f\) from \([n]\) into \(Z\), and let \(E_k\) consist of all \(f\) such that \(|f^{-1}(X)| = k\). We have \(E = \bigcup_{k=0}^{n} E_k\), \(|E| = (x+y)^n\), \(|E_k| = \binom{n}{k} x^k y^{n-k}\). Therefore

\[
(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.
\]

(4) By considering \(E\), the set of functions \(f\) from \(\{x, y, z\}\) into \([n+1]\) = \(\{1, 2, 3, \ldots, n+1\}\) such that \(f(x) < f(z), f(y) < f(z)\), and the following subsets: (i) \(E_0 = \{f \mid f(x) = k+1\}\), (ii) \(A = \{f \mid f(x) = f(y)\}\), (iii) \(B = \{f \mid f(x) < f(y)\}\), (iv) \(C = \{f \mid f(x) > f(y)\}\), we find \(E = \bigcup_{k=1}^{n} E_k\) = \(A + B + C\), i.e., with [lob]: \(|E| = \sum_{k=1}^{n} k^2 = \binom{n+1}{2} + \binom{n+1}{3} + \binom{n+1}{3}\) = \(n(n+1)(2n+1)\). (See also p. 155 and Exercise 4, p. 220.)

**Theorem A.** Let \((a_1, a_2, \ldots, a_m)\) be a sequence of \(m\) integers \(\geq 0\) such that:

\[
a_1 + a_2 + \cdots + a_m = n, \quad m \geq 1, \quad n \geq 0,
\]

then the number of divisions \(\mathcal{M} = (A_1, A_2, \ldots, A_m)\) of \(N\), \(|N| = n\) such that \(|A_1| = a_1, ie \subseteq [m]\), also called \((a_1, a_2, \ldots, a_m)\)-divisions, is equal to (note that \(0! = 1\)):

\[
\binom{n}{a_1, a_2, \ldots, a_m} \quad \text{and can be denoted by} \quad \binom{n}{a_1 a_2 \ldots a_m}
\]

or, even better, by:

\[
\binom{n}{a_1, a_2, \ldots, a_m}.
\]

Until recently one said that \(\mathcal{M}\) was a permutation with repetition of \(a_1\) elements of \(N\), \(a_2\) elements of \(N\), etc. Notation \([10c]\) which we introduce here and whose virtues we wish to recommend now, is not standard yet, but seems to become more and more in use. Anyway, it has the qualities of a good notation (cf. p. 8) and it is hard to imagine a simpler one. Moreover, it has the advantage over \([10c]\) of being coherent with the classical notation of the binomial coefficients. In fact, if we use \([10c]\) for the case of binomial coefficients, we get the notation \(\binom{n}{k} n-k\) for \(\binom{n}{k}\), which is undesirable. On the contrary, it seems good to extend the usual notation for the binomial coefficients in the case of \(\binom{x}{k}\), with \(x\) a real or complex variable, by the following notation:

\[
\binom{x}{k, x+k, \ldots, x+k} := \binom{n}{k_1, k_2, \ldots, k_j} \frac{x(x-1)(x-2)\cdots(x-k+1)}{k_1! k_2! \cdots k_j!}\]

because in this case, for \(a_1 + a_2 + \cdots + a_m = n\), we have in our notation:

\[
(a_1, a_2, \ldots, a_m) = \binom{n}{a_1, a_2, \ldots, a_m} = \binom{n}{a_1, a_2, \ldots, a_m} = \text{etc.,}
\]

which harmonizes perfectly with the binomial and multinomial notations. (This fair notation can be found in the Repertorium by [Pascal, 1910], I, p. 51.)
As \( A \) is ordered, giving \( A \) means first giving \( A_1 \), then \( A_2 \), then \( A_3 \), then \( A_4 \), etc. Now the number of possible choices for \( A_1 \subset N \), \( \mid N \mid = n \), \( \mid A_1 \mid = a_1 \), equals \( \binom{n}{a_1} \), by [5a] (p. 8). Such a choice being made, the number of possible choices for \( A_2 \subset N \setminus A_1 \), \( \mid N \setminus A_1 \mid = n - a_1 \), \( \mid A_2 \mid = a_2 \), is \( \binom{n-a_1}{a_2} \), etc. The required number (of the possible \( A \)) hence is equal to:

\[
\binom{n}{a_1} \binom{n-a_1}{a_2} \ldots \binom{n-a_1-\ldots-a_{m-1}}{a_m},
\]

which is equal to \( [10c] \) after simplification.

The notation [10a] suggests us to write \( U \setminus V \) instead of \( U \setminus V \), as in [1b] (p. 2), if \( V \subset U \). In other words, for three subsets \( U, V, W \) of \( N \):

\[ W = U - V \iff U = V + W \iff W = U \setminus V \text{ and } V \subset U. \]

The following notation also originates from [10a]:

\[ \sum_{i=1}^{m} x_i = (x_1 + x_2 + \cdots + x_m)^n = \sum (a_1, a_2, \ldots, a_m) x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m}, \]

the last summation takes place over all \( m \)-tuples \((a_1, a_2, \ldots, a_m)\) of positive or zero integers \( a_i \geq 0 \) such that \( a_1 + a_2 + \cdots + a_m = n \).

Because of this,

**Definition B.** The numbers:

\[
(a_1, a_2, \ldots, a_m) = \frac{(a_1 + a_2 + \cdots + a_m)!}{a_1! a_2! \cdots a_m!},
\]

are called multinomial coefficients.

For \( n, m \) fixed, the number of multinomial coefficients equals the number of solutions of \( a_1 + \cdots + a_m = n \) which is \( \binom{n+m-1}{n} \), by Theorems B and D (p. 15). A table of the multinomial coefficients can be found on p. 309.

We argue as in the proof of Theorem A (p. 12). Let:

\[
(x_1 + x_2 + \cdots + x_m)^n = P_1 P_2 \ldots P_n
\]

with \( P_i := x_1 + x_2 + \cdots + x_m \), the summation taking place over all systems of integers \((a_1, a_2, \ldots, a_m)\) that occur as exponents of the terms on the right-hand side of \([10g]\). Obtaining \( x_1^a x_2^b x_m^c \) in the expansion of the product \( P_1 P_2 \ldots \) is equivalent to giving a division of the set \( \{ P_1, P_2, \ldots, P_n \} \) into subsets \( A_1, A_2, \ldots, A_m \) such that \( |A_i| = a_i + \text{te}[m] \). This we do with the understanding that this division corresponds to multiplying the \( x_1 \) of the \( a_1 \) factors \( P_i \in A_1 \) by the \( x_2 \) of the \( a_2 \) factors \( P_i \in A_2 \), etc. (If \( a_1 = 0 \), then one just multiplies by 1). Hence, on one hand:

\[ a_1 + a_2 + \cdots + a_m = n, \quad a_i \in \mathbb{N}, \quad i \in [m]; \]

on the other hand, the number of terms \( x_1^a x_2^b \cdots \) where the \( a_i \) are fixed such that \([10h]\) holds, is equal to \((a_1, a_2, \ldots, a_m)\), by \([10c]\).

Thus, \( (x_1 + x_2 + \cdots + x_m)^n = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} x_1^i x_2^j x_m^k \); because the solutions of \( a_1 + \cdots + a_m = 2 \) are of the form: (I) \( a_k = 2, a_i = 0 \) if \( i \neq k \), in which case \([10c]\) = 1; (II) \( a_i = 1 \) if \( i \neq j, a_i = 0 \) if \( i = j \), in which case \([10c]\) = 2. In the same manner, \( (x_1 + x_2 + \cdots + x_m)^n \) is exactly \( n(n) \), the number of partitions of \( n \), p. 94. (See also Exercise 28, p. 126, and Exercise 9, p. 158.) Multinomial coefficients enjoy congruence properties, analogous to [6g, g'] p. 14, the proof being very similar:

**Theorem C.** For any prime number \( p \) and \( a_1 + a_2 + a_3 + \cdots = p \), we have

\[
(a_1, a_2, a_3, \ldots) \equiv 0 \pmod{p},
\]

except \((p, 0, 0, \ldots) = (0, p, 0, \ldots) = \cdots = 1.\)

In other words, for variables \( x_1, x_2, \ldots, x_m \),

\[
(x_1 + x_2 + \cdots + x_m)^n \equiv x_1^n + x_2^n + \cdots + x_m^n.
\]
Definition C. A non-ordered (finite) set \( P \) of \( p \) blocks of \( N \) (\( \equiv p \)-system of \( N \), cf. p. 3), \( P \subseteq \mathcal{P}(N) \), is called a partition of \( N \), or \( p \)-partition if one wants to specify the number of its blocks, if the union of all blocks of \( P \) equals \( N \), and if these blocks are mutually disjoint.

Hence in a partition, as opposed to a division (1) no ‘subset’ is empty; (2) the ‘subsets’ are not labelled.

Similar to \([10a]\), we denote for such a partition, in order to express the fact that \( B, B' \in \mathcal{P} \Rightarrow B \cap B' = \emptyset \):

\[
N = \sum_{B \in \mathcal{P}} B, \quad \forall B \in \mathcal{P}, \quad |B| > 1.
\]

Evidently there is a bijection between the set of equivalence relations of \( N \) and the set of partitions of \( N \); we just associate with every equivalence relation \( \mathcal{R} \) the partition whose blocks are the equivalence classes of \( \mathcal{R} \).

Theorem D. Let \( f \) be a map of \( M \) into \( N, f \in N^M \). The set of the nonempty pre-images \( f^{-1}(y), y \in N \) (p. 5) constitutes a partition of \( M \), which is called the partition induced by \( f \) on \( M \).

This is evident. It follows in particular, for each \( f \in N^M \) that:

\[
|M| = \sum_{y \in N} |f^{-1}(y)|.
\]

1.11. Bound variables

It is well known that a finite sum of \( n \) terms \( x_1, x_2, \ldots, x_n \), real numbers, or, more generally, in a ring, is denoted by \( x_1 + x_2 + \cdots + x_n \) (such a way of writing, of course, does not mean at all that \( n \geq 3 \)), or even better:

\[
\sum_{k=1}^n x_k.
\]

We generalize this notation. Let \( m \) be an integer \( \geq 1 \), and \( f \) a real-valued function (or, more generally, with values in a ring) defined for all points (\( =m\)-tuples) \( c := (c_1, c_2, \ldots, c_m) \) of a product set:

\[
E := E_1 \times E_2 \times \cdots \times E_m.
\]

(Frequently we will have \( E_1 = E_2 = \cdots = E_m = N \).) If \( f \) is only defined on \( \Omega(=E) \), it will be extended to the whole of \( E \) by 0, in most cases. Let us consider a finite set \( \Gamma \subseteq E \). The expression \( S \), denoted in any of the following four ways:

\[
[S = \sum_{c \in \Gamma} f(c) = \sum_{(c_1, c_2, \ldots, c_m) \in \Gamma} f(c_1, c_2, \ldots, c_m) = \sum_{(c_1, c_2, \ldots, c_m) \in \Gamma} f(c_1, c_2, \ldots, c_m) = \sum_{(c_1, c_2, \ldots, c_m) \in \Gamma} f(c_1, c_2, \ldots, c_m)
\]

equals by definition the finite sum of the values of \( f \) in each point \( c \) of \( \Gamma \), which is called the summation set. If \( \Gamma \cap E = \emptyset \), we give \( S \) the value 0.

\[
[11d] \quad \text{Empty sum convention: } \sum_{c \in \emptyset} f(c) = 0.
\]

Sometimes we qualify \( S \) by saying that it is a multiple sum of order \( m \). For \( m = 1, 2, 3, \ldots \), one says usually simple, double or triple sum.

It is clear that the value \([11c] \) of \( S \) is completely determined by \( \Gamma \) and \( f \). Thus, \( S \) does not depend on \( c := (c_1, c_2, \ldots, c_m) \), even though it occurs in formula \([11c] \). For this reason, the letters \( c \) or \( (c_1, c_2, \ldots, c_m) \) are called bound variables of the summation (dummy or dead are also used synonymously for bound). It is useful to note the analogy with the notation \( \int_a^b f(x) \, dx \) of the integral, in which \( x \) is also a bound (real) variable, while \( I \) only depends on \( a, b \) and \( f \).

Usually, the summation set \( \Gamma \) is defined by a certain number of conditions or restrictions, \( \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_l \) on the \( c_1, c_2, \ldots, c_m \); these conditions will just be translated by saying that the point \( c \) belongs to the subsets \( \Gamma_1, \Gamma_2, \ldots, \Gamma_l \). We will therefore write any of the following:

\[
[S = \sum_{x_1, x_2, \ldots, x_l} f(c) = \sum_{x_1, x_2, \ldots, x_l} f(c) = \sum_{x_1, x_2, \ldots, x_l} f(c) = \sum_{x_1, x_2, \ldots, x_l} f(c) = \sum_{x_1, x_2, \ldots, x_l} f(c) = \sum_{x_1, x_2, \ldots, x_l} f(c) = \sum_{x_1, x_2, \ldots, x_l} f(c)
\]

For example, \([11e] \) is equivalent with \([11a] \):

\[
[S = \sum_{1 \leq k \leq n} x_k \quad \text{or} \quad \sum_{1 \leq k \leq n} x_k.
\]

If the expression for the \( \mathcal{C}_j \) is not very simple, it is better to avoid writing it underneath or on the side of the summation sign \( \sum \); but following it. In that case one uses a phrase like “the summation takes place over all \( c \) such that ...”.

Quite often one needs some letters different from \( c_1, c_2, \ldots, \) say \( d_1, d_2, \ldots, \)
in the detailed description of the conditions \( \mathcal{C}_i \). It is important to distinguish these from the bound variables, especially in the case that we wish to use notation \([11e]\). Therefore we introduce the

\[ \text{[11g]} \quad \text{DOT CONVENTION: every letter with a dot underneath stands for a bound variable.} \]

Of course, we do not have to dot every bound variable: in \([11f]\), for example, there is but one possible interpretation. We must try to limit the dots to the cases where there is possible danger of confusion or ambiguity (examples follow). Furthermore, each variable needs only to be pointed once, and not every time it appears in the conditions \( \mathcal{C}_1, \mathcal{C}_2, \ldots \). In general, however, we are not at all embarrassed by excesses, as far as this is concerned. The use of dots under the bound variables is imposed upon us by our total and absolute rejection of the notation by repeated \( \Sigma \) signs (which is still commonly used), for any multiple sum of order \( m \) (Theorem B below).

Before demonstrating the preceding by examples, we still put the

\[ \text{[11h]} \quad \text{NONNEGATIVE INTEGER CONVENTION: in the sequel of this book each bound variable will represent an integer \( \geq 0 \) unless stated otherwise.} \]

Now we give the following results:

**Theorem A (associativity).** For all partitions \( \mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \) of \( \Gamma = \mathcal{A}_1 + \mathcal{A}_2 + \cdots + \mathcal{A}_n \), we have:

\[ \text{[11i]} \quad S := \sum_{c \in \mathcal{A}} f(c) = \sum_{c_1 \in E_1} \left( \sum_{c_2 \in E_2} f(c_1, c_2) \right). \]

**Theorem B (analogue of the Fubini theorem for multiple integrals):**

\[ \text{[11j]} \quad \sum_{(c_1, c_2) \in E_1 \times E_2} f(c_1, c_2) = \sum_{c_1 \in E_1} \left( \sum_{c_2 \in E_2} f(c_1, c_2) \right). \]

\[ \text{[11k]} \quad \sum_{(c_1, c_2, c_3) \in E_1 \times E_2 \times E_3} f(c_1, c_2, c_3) = \sum_{c_1 \in E_1} \left( \sum_{c_2 \in E_2} \left( \sum_{c_3 \in E_3} f(c_1, c_2, c_3) \right) \right). \]

(For the number of possible 'Fubini formulas' see Exercise 20 on p. 228.)

**Examples.** (I) To calculate, for \( n \geq 0 \) integer, the double sum:

\[ S := \sum_{c_1 + c_2 = n} c_1 c_2. \]

We get, if we reduce it to a simple sum:

\[ S = \sum_{0 \leq c_1 \leq n} c_1 (n - c_1) = n \sum_{0 \leq c_1 \leq n} c_1 - \sum_{0 \leq c_1 \leq n} c_1^2 \]

\[ = n \frac{(n + 1)(2n + 1)}{2} - \frac{n(n + 1)(2n + 1) - n(n^2 - 1)}{6}. \]

(See also Exercise 28 on p. 85 for a generalization.)

(II) To calculate, for \( a \) and \( b \) complex, \( a, b, ab \neq 1 \) and \( n \) an integer \( \geq 0 \), the double sum:

\[ S := \sum_{0 \leq k' \leq k \leq n} a^k b^k. \]

We can do this as follows, where we use Theorem B for the equality (*)

\[ S = \sum_{0 \leq k' \leq k \leq n} (a^k \sum_{k' \leq k \leq n} b^k) = \sum_{0 \leq k' \leq k \leq n} a^k \frac{b^{k+1} - b^k}{b - 1} \]

\[ = \frac{b^{k+1}}{b - 1} \sum_{0 \leq k' \leq k \leq n} a^k - \frac{1}{b - 1} \sum_{0 \leq k' \leq k \leq n} (ab)^k \]

\[ = \frac{b^{k+1} (a^{k+1} - 1)}{(b - 1)(a - 1)} - \frac{(ab)^{k+1} - 1}{(b - 1)(a - 1)}. \]

We could also have started with \( S = \sum_{0 \leq k' \leq n} (b^k \sum_{0 \leq k' \leq n} a^k). \)

(III) For any finite set \( N \), \( |N| = n \), to calculate the double sum:

\[ S := \sum_{A \subseteq N} |A \cap B|. \]

(The summation is taken over all pairs of subsets \( (A, B) \in \mathcal{P}(N) \times \mathcal{P}(N) \).)

By Theorem B, we get for \( S \):

\[ \sum_{A \subseteq N} \left( \sum_{B \subseteq N} |A \cap B| \right) = \sum_{A \subseteq N} \left( \sum_{0 \leq |A| \leq |N|} |A \cap B| \right). \]

Now it is easy to see, that the number of subsets \( B(\subseteq N) \) such that
\[ |A \cap B| = i, \text{ where } A \text{ is fixed, equals } \frac{|A|!}{i!} \cdot 2^{n-|A|}, \] which is the number of \( i \)-subsets of \( A \) times the number of subsets of \( N-A \). Hence, as \( \sum_{i=0}^{|A|} \binom{|A|}{i} = 2^{n-1} \) (which results from taking the derivative of the polynomial \((1+x)^n = \sum_{i=0}^n \binom{n}{i} x^i\), and substituting 1 for \( x \)), which we use for equality (*), we get for \( S \):

\[
\sum_{A \subseteq N} \left( 2^{n-|A|} \sum_{0 \leq i \leq |A|} \binom{|A|}{i} \right) = \sum_{A \subseteq N} 2^{n-|A|} 2^{|A|-1} = 2^{n-1} \sum_{A \subseteq N} |A| = 2^{n-1}, \ n2^{n-1} = n4^{n-1}.
\]

More symmetrically, we could have said also:

\[
S = \sum_{\delta \subseteq N} \left( \sum_{A \subseteq B} |A \cap B| \right) = \sum_{K \subseteq N} 3^{n-|K|} |K| = \sum_{0 \leq k \leq n} \binom{n}{k} 3^{n-k} k = n4^{n-1}.
\]

(Furthermore, \( \sum |A_1 \cap \cdots \cap A_k| = n2^{k(n-1)}, \) where \( A_1, A_2, \ldots, A_k \subseteq N. \))

In certain cases, we can immediately lower the order of a summation by applying Theorems A and B:

**Theorem C.** If \( f(c_1, c_2, \ldots, c_m) = f_1(c_1, c_2, \ldots, c_n) f_2(c_{n+1}, \ldots, c_m), \) \( 0 < h < m, \) then:

\[
\sum_{(c_1, c_2, \ldots, c_m) \in E_1 \times \cdots \times E_m} f(c_1, c_2, \ldots, c_m) = \left( \sum_{(c_1, \ldots, c_n) \in E_1 \times \cdots \times E_n} f_1(c_1, \ldots, c_n) \right) \times \left( \sum_{(c_{n+1}, \ldots, c_m) \in E_{n+1} \times \cdots \times E_m} f_2(c_{n+1}, \ldots, c_m) \right).
\]

Particularly:

\[
\sum_{(c_1, \ldots, c_n) \in E_1 \times \cdots \times E_m} g_1(c_1) \ldots g_m(c_m) = \left( \sum_{c_1 \in E_1} g_1(c_1) \right) \left( \sum_{c_2 \in E_2} g_2(c_2) \right) \ldots \left( \sum_{c_m \in E_m} g_m(c_m) \right).
\]

It will be noticed that this theorem bears some analogy to the theorem on double integrals: if \( A = [a, b] \times [c, d] \) then \( \int_A f(x) g(y) \, dx \, dy = (\int_c^d f(x) \, dx) (\int_a^b g(y) \, dy). \)

Clearly, everything that has been said in this section about the notation of finite sums, can be repeated, with the necessary changes, for any expression in which addition is replaced by an internal associative and commutative composition law in the range of \( f \). Thus, we denote:

\[
\prod_{1 \leq k \leq n} x_k \quad \text{for the product } \quad x_1 x_2 \ldots x_n.
\]

\[
\bigcup_{1 \leq k \leq n} A_k \quad \text{for the union } \quad A_1 \cup A_2 \cup \cdots \cup A_n;
\]

\[
\bigcap_{1 \leq k \leq n} A_k \quad \text{for the intersection } \quad A_1 \cap A_2 \cap \cdots \cap A_n.
\]

Conventions [11g, h] still hold for \( \prod, \cup, \cap, \) but [11d] (p. 31) is replaced by [11n, o, p]:

\[
[11n] \quad \text{EMPTY PRODUCT CONVENTION: } \prod_{c \in A} f(c) = 1.
\]

\[
[11o] \quad \text{EMPTY UNION CONVENTION: } \bigcup_{c \in A} (c) = \emptyset, \text{ where } A \subseteq N.
\]

\[
[11p] \quad \text{EMPTY INTERSECTION CONVENTION: } \bigcap_{c \in A} (c) = N, \text{ where } A \subseteq N.
\]

**Example.** Compute, for \( n \) integer \( \geq 1 \), the double product:

\[
P := \prod_{p+q \leq n} a^p b^q.
\]

We can work this out as follows, using [5g] on p. 10 for (*):

\[
P = \prod_{0 \leq k \leq n} \left( \prod_{0 \leq q \leq k} a^p b^q \right) = \prod_{0 \leq k \leq n} \left( \sum_{0 \leq q \leq k} a^p b^{k-q} \right) = \prod_{0 \leq q \leq k} \left( \sum_{0 \leq p \leq n} \binom{n}{p} a^p b^{k+1/2} \binom{k+1/2}{2} \right)
\]

More generally, it can be found without difficulty that the \( i \)-th order product \( \prod_{j \leq n} a^j \) has the value \((a_1, a_2, \ldots, a_i)^q\) with \( q = \binom{i+n}{i+1}.\)
1.12. Formal series

(1) General remarks

The concept of formal power series is a generalization of polynomial. We think the best is to sketch here the outlines of the theory, following Bourbaki ([*Bourbaki, Algèbre*, chap. 4, 5, 1959], p. 52–69; see also [*Dubreil (P. and M.-L.), 1964*], p. 124–31, [*Lang, 1965*], p. 146, [*Zariski, Samuel, II, 1960*], p. 129); we will refer to this author for proofs and more details.

In this section, each small Greek letter represents a finite sequence of k integers \( \geq 0 \), where \( k \) is an integer \( \geq 1 \), which is given once and for all. Such a sequence is sometimes also called a multi-index. Thus, if we write \( \kappa := N^\kappa \), in which \( \kappa := \{1, 2, \ldots, k\} \), then \( x \in \kappa \) means that \( x = (x_1, x_2, \ldots, a_k) \), where \( x_i \in \mathbb{N} \).

We may denote:

\[ [x] := a_1! a_2! \cdots a_k! \]
\[ |x| := a_1 + a_2 + \cdots + a_k, \]
\[ c_x := c_{a_1 a_2 \cdots a_k}, \]
\[ \ell^x := t_1^{a_1} t_2^{a_2} \cdots t_k^{a_k}. \]

We will consider the case of formal series in \( k \) variables over a field \( C \) (often \( C = \mathbb{R} \) or \( C = \mathbb{C} \)).

**Definition A.** A formal power series \( f \) in \( k \) indeterminates \( (or variables) \) \( t_1, t_2, \ldots, t_k \) over \( C \) is a formal expression of the following type:

\[ f = f(t) = f(t_1, t_2, \ldots, t_k) = \sum_{m_1, m_2, \ldots, m_k \geq 0} a_{m_1 m_2 \cdots m_k} t_1^{m_1} t_2^{m_2} \cdots t_k^{m_k}, \]

where \( a_{m_1 m_2 \cdots m_k} \), the coefficients of \( f \), form a multiple series of order \( k \) with values in \( C \). Each expression \( a_{m_1 m_2 \cdots m_k} t_1^{m_1} t_2^{m_2} \cdots t_k^{m_k} \) is called a monomial of \( f \). As the \( m_1, m_2, \ldots, m_k \) are bound variables, they can have a dot underneath. We denote \( C[[t_1, t_2, \ldots, t_k]] \), or even better \( C_k[[t]] \), which is called the set of formal series \( f \).

\( f \) is a polynomial if all coefficients except a finite number of them equal zero, which is usually formulated by saying "almost all \( a_{m_1 m_2 \cdots m_k} \) are zero". In simple cases we sometimes avoid to write [12b] by using an *ellipsis* mark, three consecutive periods, especially if there is only one indeterminate. For example:

\[ f = 1 + t + t^2 + \cdots = \sum_{n \geq 0} t^n \in \mathbb{R}, \quad f = f(t) = f(t_1, t_2, \ldots) = \sum_{n \geq 0} t^n \in \mathbb{R}. \]

Every power series in several variables, which is convergent in a certain polydisc, can be interpreted as a formal series. Conversely, with every formal series in several indeterminates can be associated with a power series that perhaps converges in the point \( 0 \) only. The following expansions:

\[ [12c] \quad \exp t := \sum_{n \geq 0} \frac{t^n}{n!}, \]
\[ [12d] \quad \log(1 + t) := \sum_{n \geq 1} (-1)^{n+1} \frac{t^n}{n}, \]
\[ [12e] \quad (1 + t)^x := \sum_{n \geq 0} \binom{x}{n} \frac{t^n}{n!} \quad (x \in C), \]
\[ [12e'] \quad (1 - t)^{-x} := \sum_{n \geq 0} \frac{(x^n)}{n!} \quad (x \in C), \]
\[ [12f] \quad (1 - t)^{-x} := \sum_{n \geq 0} \frac{(x^n)}{n!} \quad (x \in C), \]
\[ [12g] \quad (1 - t)^{-x} := \sum_{n \geq 0} \frac{(x^n)}{n!} \quad (x \in C). \]

can be as well considered as functions in their radius of convergence as well as certain formal series, which are called respectively: *formal exponential series, formal logarithm, formal binomial series* (of the 1st and 2nd form). Moreover, for [12c] we have also, if \( x \) is an integer \( > 1 \):

\[ (1 - t)^{-x} = \sum_{n \geq 0} \binom{n + x - 1}{x - 1} t^n. \]

Furthermore, the series [12e, e'] can also be interpreted as series in two indeterminates \( t \) and \( x \).

*From now on, in the sequel of this book, each power series must be considered as a formal series, unless explicitly stated otherwise.*

As in the case of polynomials, \( C_k[[t]] \) becomes an integral domain, if we provide it with addition and multiplication as follows: for every \( f = \sum a_n t^n \) and \( g = \sum b_n t^n \) where \( m \in \kappa \):

\[ [12f] \quad f + g := \sum_{n \in \kappa} c_n t^n, \quad c_n := a_n + b_n, \]
\[ [12g] \quad fg := \sum_{n \in \kappa} d_n t^n, \]
where

\[ d_{\mu} = d_{\mu_1, \ldots, \mu_k} - \sum_{\sum_{i=1}^k x_i = \mu} a_{x_1 \ldots x_k} b_{\lambda_1 \ldots \lambda_k}, \]

the last summation taken over all sequences of integers \( \geq 0 \), \((x_1, \ldots, x_k, \lambda_1, \ldots, \lambda_k)\) such that \( x_1 + \lambda_1 = \mu_1, \ldots, x_k + \lambda_k = \mu_k \) (hence we have \((\mu_1 + 1) \ldots (\mu_k + 1)\) terms in the last summation).

The \textit{homogeneous part} of \( f \) of degree \( m \) is the formal polynomial:

\[ f_{(m)} := \sum_{|g| = m} a_{g} t^{g} = \sum_{g_1 + \cdots + g_n = m} a_{g_1, \ldots, g_n} t^{g_1} \cdots t^{g_n}. \]

The constant term of \( f \) is \( a_0 = f(0) \), also denoted by \( f(0) \). The order of \( f \) (which we suppose different from the series 0, all whose coefficients equal zero), is the smallest integer \( n \geq 0 \), such that \( f(n) \neq 0 \). For example, \( \omega(t_1 t_2 + (t_1 t_2)^2 + \cdots) = 2 \). Clearly, \( \omega(f g) = \omega(f) + \omega(g) \). The series 1 is the series all whose terms are zero except the constant term, which equals 1.

We give two examples. (I) The family

\[ f_{l_1, l_2} := \sum_{\mu_1, \mu_2 > 0} l_1^{(\mu_1 + 1)} l_2^{(\mu_2 + 1)} = \sum_{\mu_1 \neq 0} l_1^{(\mu_1 + 1)} \sum_{\mu_2 \neq 0} l_2^{(\mu_2 + 1)} = l_1^{l_1 l_2} (1 - l_1^{l_1})^{-1} (1 - l_2^{l_2})^{-1} \]

is summable, \((l_1, l_2) \in \mathbb{N}^2 \). In the definition of \( f_{l_1, l_2} \) the exponents \( l_1 (\mu_1 + 1) \) and \( l_2 (\mu_2 + 1) \) are replaced by \( l_1 \mu_1 \) and \( l_2 \mu_2 \), then the family is not summable anymore. (2) The family \( f_{C_n} \) of homogeneous parts of \( f \), \([12h]\), is summable, and \( f = \sum_{\mu > 0} f_{(\mu)} \). Moreover, we have the 'Cauchy product' form for the series \( h \), which is the product of \( f \) and \( g \):

\[ [12j] \quad h = fg \iff h_{(n)} = \sum_{0 \leq i \leq n} f_{(i)} g_{(n-i)}. \]

**Theorem A (associativity).** Let be given a summable family of formal series, \((f_1)_{i \in \mathbb{N}}\), with sum \( g \), and \((L_i)_{i \in \mathbb{N}}\) a division \((p, q)\), possibly infinite, of \( L = \sum_{i \geq 1} L_i \), then every subfamily \((f_1)_{i \in L_i}\) is summable, and we have \( g = \sum_{i \in L} f_i \). Moreover, we have the 'Cauchy product' form for the series \( h \), which is the product of \( f \) and \( g \):

**Theorem B (products).** Let \((f_1)_{i \in \mathbb{N}}\) and \((g_m)_{m \in \mathbb{N}}\) be two summable families. Then the family \((f_1 g_m)_{i \in \mathbb{N}, m \in \mathbb{M}}\) is summable, and we have \( \sum_{i \in \mathbb{N}} f_1 g_m = \sum_{i \in \mathbb{N}} f_i \sum_{m \in \mathbb{M}} g_m = \sum_{i \in \mathbb{N}} f_i \sum_{m \in \mathbb{M}} g_m \).

The generalization to a finite product is evident.

(III) \underline{Multiplicable families of formal series}

**Definition C.** A family of formal series \((f_1)_{i \in \mathbb{N}}\) is called multiplicable if for each sequence \( \mu \in \mathbb{K} \), the coefficient \( a_{1, \mu} \) of \( t^\mu \) in \( f_1 \) equals 0 for almost all \( i \in \mathbb{N} \) (except for a finite number, see p. 36). The sum \( g = \sum_{i \in \mathbb{N}} g_{(i)} \) is then defined by:

\[ b_{\mu} := \text{the coefficient of } t^\mu \text{ in the finite sum } \sum f_1, \text{ where } i \in \mathbb{L}, \text{ and } \omega(f_1) \leq |\mu|. \]

We denote \( g = \prod_{i \in \mathbb{L}} f_i \).

For \( \mathbb{L} = \mathbb{N} \), \((f_i)\) is evidently multiplicable if and only if the order \( \omega(f_i) \) tends to infinity, when \( i \) tends to infinity.
when \( t \) tends to infinity. For example, \( f_i := (1 + t_i t_j) \) is multiplicable. Every finite family is evidently multiplicable, and we get back definition \([12g]\) for the product. Explicitly, for one single variable \( t \) and one sequence \( \{f_i\} \) of formal series, \( i = 1, 2, \ldots, f_i := \sum_{n \geq 0} a_{i,n} t^n \), we have, if we write out the bound variables \( n_i \) completely in \((*)\):

\[
\prod_{i \geq 1} f_i = \prod_{i \geq 1} \left( \sum_{n_i \geq 0} a_{i,n_i} t^{n_i} \right) = \sum_{n_1, n_2, \ldots \geq 0} a_{1,n_1} a_{2,n_2} \cdots t^{n_1 + n_2 + \cdots} = \sum_{n \geq 0} \left( \sum_{n_1 + n_2 + \cdots = n} a_{1,n_1} a_{2,n_2} \cdots \right) t^n,
\]

where the last summation makes sense, because it contains only a finite number of terms (cf. Definition C). (On this subject, see also p. 130.)

(VI) Substitution (also called composition) of formal series

**Theorem C.** Let \( \{g_i\}_{i \in \mathbb{P}} \) be \( p \) formal series \( \in C_q[[t]] \) without constant terms: \( \sigma(g_i) \geq 1 \). We can 'substitute' \( g_i \) for \( u_i \), \( i \in \mathbb{P} \), into every formal series \( f = \sum_{n \geq 0} a_n u^n \in C_q[[u]] \). In this way we obtain a new formal series, called the composition of \( f \) and \( g \), and denoted \( f \circ g \), which belongs again to \( C_q[[t]] \). By definition, \( f \circ g \) equals the sum of the summable family \( a_{1, \ldots, p} (g_1)^{\mu_1} \cdots (g_p)^{\mu_p} \), where \( \mu = (\mu_1, \mu_2, \ldots, \mu_p) \in \mathbb{P} \).

For example, using \([12c, d]\), it can be verified that

\[
\log(1 + t) = t, \quad \exp(\log(1 + t)) = 1 + t.
\]

Now we want to find the formal expansion of \( h := (1 + t_1 + t_2 + \cdots + t_q) \in R_q[[t]] \). Applying Theorem C, with \( f := (1 + u)^{x} \in R_q[[u]] \), \( g := t_1 + t_2 + \cdots + t_q \in R_q[[t]] \), we get by using \([12c]\) (p. 37) for equality \((*)\) and \([10f]\) (p. 28) for \((**):\)

\[
h = (**) \sum_{n \geq 0} \left( \sum_{n_1 + \cdots + n_q = n} \frac{n!}{n_1! \cdots n_q!} t_1^{n_1} \cdots t_q^{n_q} \right) = (**) \sum_{n \geq 0} \frac{t^n}{1 + \cdots + t_q}.
\]

which gives after simplifications:

\[
[12m] \quad \left( 1 + t_1 + t_2 + \cdots + t_q \right)^x = \sum_{v_1, \ldots, v_q \geq 0} \frac{t_1^{v_1} t_2^{v_2} \cdots t_q^{v_q}}{v_1! v_2! \cdots v_q!},
\]

using an evident extension of the notations \([7b]\) (p. 16) and \([10c]\) (p. 27).

We can also establish, using multinomial coefficients \((v) := (v_1, v_2, \ldots, v_q)\) of \([10c]\) (p. 27), the corresponding expansions for \( \log: \)

\[
\log(1 + t_1 + t_2 + \cdots + t_q) = \sum_{v_1, \ldots, v_q \geq 0} (v_1 + \cdots + v_q) t_1^{v_1} t_2^{v_2} \cdots t_q^{v_q}
\]

using evident extension of the notations \([7b]\) (p. 16) and \([10c]\) (p. 27).

(V) Transformations of formal series

With every formal series \( f = \sum_{n \geq 0} a_n t^n \) in one indeterminate \( t \), we can associate the formal derivative, denoted by:

\[
Df = \frac{df}{dt} = \sum_{n \geq 0} n a_n t^{n-1} = \sum_{n \geq 0} (n + 1) a_{n+1} t^n,
\]
and also the formal primitive:

$$Pf = \int_0^t f(u) \, du = \sum_{n \geq 0} \frac{a_n^{p+1}}{n+1}.$$ 

All the usual properties hold: $DPf = f$, $D(fg) = (Df) \cdot g + f \cdot (Dg)$, etc.

The iterates of these operations can easily be found. For the derivation we have:

$$D^k f = \sum_{\eta, \kappa} (\eta)_\kappa a_\eta t^{\eta-k} = \sum_{\eta, \kappa} \langle \eta+1 \rangle_\kappa \, t^{\eta},$$

and for the primitivation we have:

$$P^k f = \sum_{\eta, \kappa} a_\eta \frac{t^{\eta+k}}{(n+1)_{\kappa}} = \sum_{\eta, \kappa} a_{\eta-k} \frac{t^{\eta}}{(m-k)_{\kappa}} \int_0^t \frac{(t-x)^{k-1}}{(k-1)!} \, f(x) \, dx.$$ 

These concepts can be generalized without difficulties to more indeterminates. For example, for $f = \sum_{\alpha, \beta} a_\alpha t^\beta$, we define:

$$D^\alpha f = \frac{\partial^{\alpha_1} \partial^{\alpha_2} \ldots \partial^{\alpha_n}}{\partial t_1^{\alpha_1} \partial t_2^{\alpha_2} \ldots \partial t_n^{\alpha_n}} f(t_1, t_2, \ldots, t_n)$$

$$= \sum_{\gamma_1, \ldots, \gamma_n \geq 0} (\gamma_1)_1 \ldots (\gamma_n)_n a_{\gamma_1, \ldots, \gamma_n} t_1^{\gamma_1} \ldots t_n^{\gamma_n}.$$ 

We mention here also the transformation that associates to every double series $f(x, y) = \sum_{m, n \geq 0} a_{m,n} x^m y^n$ its diagonal series $\varphi(t) = \sum_{n \geq 0} a_{n,n} t^n$. When $f(x, y)$ converges, we have ([Hautus, Klarner, 1971]):

$$\varphi(t) = \frac{1}{2\pi i} \int_{|z|=\varepsilon} f(tz/z) \, dz,$$

where $\varepsilon$ and $|t|$ are sufficiently small, so that $f(x, y)$ is regular for $|x| < \varepsilon$ and $|y| < |t|/\varepsilon$. In general, it is tantamount to saying that the circle $|z|=\varepsilon$ contains all the poles of $f(z, t/z)$ that tend to 0 when $t$ tends to 0. For instance, for $f(x, y) = \sum_{m, n} (m, n) x^m y^n = (1 - x - y)^{-1}$, where the $(m, n) = \binom{m+n}{m}$ are the binomial coefficients in the symmetrical notation (p. 8), the diagonal $\varphi(t) = \sum_{n \geq 0} \binom{2n}{n} t^n$ equals the residue of $(1 - z - t/z)^{-1}$ in the point $z = (1 - (1 - 4t)^{1/2})/2$, in other words $(1 - 4t)^{-(1/2)}$. This result is of course well-known (see Exercise 22 (1), p. 81).

(VI) Formal Laurent series

These series are written analogously to the preceding, [12b] (p. 36), but here the indices and the exponents $\mu_1, \mu_2, \ldots, \mu_k$ can take all integer values $\geq 0$, with the condition that the coefficients $a_{\mu_1, \ldots, \mu_k}$ that contain at least one index $< 0$, are almost all zero. For example:

1. With one single indeterminate $t$: $(t^2 + t^3 + \ldots)^{-1} = (t^2 (1 - t)^{-1})^{-1} = t^{-2} - t^{-1}$.

2. With two indeterminates $t_1$ and $t_2$: $\sum t_1^{\mu_1} t_2^{\mu_2}, \mu_1 \leq \mu_2 \leq 2\mu_1 + 10$, where the integers $\mu_1, \mu_2$ can be negative as well as positive or zero.

All the preceding: operations, summable families, derivation, etc., can be easily done for such series.

(VII) Formal series in 'noncommutative' indeterminates ([Schützenberger, 1961])

Let $X^*$ stand for the free monoid generated by $X$ (see p. 18) and let $f: \mu \mapsto a_{\mu}$ be a map from $X^*$ into a certain ring $A$ ($\mu$ is a word over $X$). If we write $f$ as a formal series: $f := \sum_{\mu \in X^*} a_\mu \mu$, then the set $A^X$ of these maps $f$ becomes an algebra, called the monoid algebra $A^X$ if, for $g := \sum_{\mu \in X^*} b_\mu \mu$, we put $f + g := \sum_{\mu \in X^*} (a_\mu + b_\mu) \mu$ and $fg := \sum_{\mu \in X^*} c_\mu \mu$, where $c_\mu = \sum a_\mu b_\mu$, the finite summation being taken over all pairs $(\mu, \lambda)$ of words such that $x \lambda = \mu$, in the sense of the juxtaposition product of p. 18. If $X$ is finite and if one considers the Abelian words of $X$, then the ordinary formal series studied above are found back again.

1.13. Generating functions (abbreviated GF)

(1) Simple sequences

Definition. Let be given a real or complex sequence (in this book actually often consisting of positive integers with a combinatorial meaning), then we call ordinary GF, exponential GF, and more generally, GF according
to Ω, of the sequence aₙ the following three formal series Ψ, Φ and Φₐ, respectively, where Ωₐ is a fixed given sequence:

[13a] \[ Φ(t) := \sum_{n≥0} aₙ tⁿ, \quad Ψ(t) := \sum_{n≥0} aₙ tⁿ/n!, \quad Φₐ(t) := \sum_{n≥0} Ωₐₙ tⁿ. \]

The most interesting case is that where (at least) one of the entire series [13a] has a positive nonzero radius of convergence R, and converges for |t|<R to a composition of elementary known functions; in this case the properties of these functions can be used to give new information about the aₙ. (For a detailed study of the relation between aₙ and their GF, the reader is referred to any work on difference calculus; for example [Jordan (Ch.), 1947] or [Milne-Thomson, 1933].)

Example A. aₙ := \( \binom{x}{n} \), where x∈R or C. Then \( Φ(t) = \sum_{n≥0} \binom{x}{n} tⁿ = \sum_{n≥0} (x)_n tⁿ/n! = (1+t)^x \), which converges for |t|<1 (if t∈C one chooses the value of Q(t) that equals 1 for t=0). If we compare the coefficients of \( tⁿ/n! \) in the first and the last member of equalities [13b]:

[13b] \[ \sum_{n≥0} (x+y)_n tⁿ/n! = (1+t)^x (1+t)^y = \left( \sum_{k≥0} \binom{x}{k} tᵏ \right) \left( \sum_{k≥0} \binom{y}{k} tᵏ \right) = \sum_{n≥0} \left( \sum_{k≥0} \binom{x}{k} \binom{y}{n-k} \right) tⁿ/n! \]

we obtain the Vandermonde convolution, in two forms:

[13c] \[ (x+y)_n = \sum_{0≤k≤n} \binom{n}{k} (x)_k (y)_{n-k}, \]

[13c'] \[ \binom{x+y}{n} = \sum_{0≤k≤n} \binom{x}{k} \binom{y}{n-k}. \]

(see also p. 26). Similarly, one shows, using \( \sum_{n≥0} \binom{x}{n} (tⁿ/n!)(1-t)^{-x} = (1-t)^{-x} \):

[13d] \[ \langle x+y \rangle_n = \sum_{0≤k≤n} \binom{n}{k} \langle x \rangle_k \langle y \rangle_{n-k}, \]

[13d'] \[ \binom{x+y}{n} = \sum_{0≤k≤n} \binom{x}{k} \binom{y}{n-k}. \]

Example B. Fibonacci numbers. These are integers Fₙ defined by:

[13e] \[ Fₙ = Fₙ₋₁ + Fₙ₋₂, \quad n ≥ 2; \quad F₀ = F₁ = 1. \]

We want to find the ordinary GF, \( Φ = \sum_{n≥0} Fₙ tⁿ \):

[13f] \[ Φ = 1 + t + \sum_{n≥2} (Fₙ₋₁ + Fₙ₋₂) tⁿ = 1 + t Φ + t² Φ. \]

Comparing the first and the last member of these equalities we obtain:

[13g] \[ Φ = \frac{1}{1-t-t²}. \]

If we decompose this rational function into partial fractions, putting the roots of \( 1-t-t² = 0 \) equal to -α, -β, we get:

[13h] \[ \frac{1}{1-t-t²} = \frac{1}{\sqrt{5}} \left( \frac{β^{n+1} - α^{n+1}}{β-α} \right). \]

Hence, identifying the coefficients of \( tⁿ \) in [13f, g]:

[13i] \[ Fₙ = \frac{β^{n+1} - α^{n+1}}{\sqrt{5}}, \]

where

\[ α = \frac{1-\sqrt{5}}{2}, \quad β = \frac{1+\sqrt{5}}{2}. \]

(One can take also as initial conditions \( F₀=0, F₁=1 \) [Hardy, Wright, 1965], p. 148, in which case \( Φ(t) = (1-t-t²)^{-1} \) and \( Fₙ = (βⁿ-αⁿ)/\sqrt{5}. \))

Here we find the golden ratio, \( β = 1.61803... \) of the Renaissance architects.

Moreover, if we let \( \|x\| \) denote the integer closest to x (x not supposed to be half-integral), then [13h] shows easily that \( Fₙ = \|β^{n+1}/\sqrt{5}\| \).

The Fibonacci numbers have a simple combinatorial meaning: \( Fₙ₊₁ \) is the number of subsets of \( \{n\} = \{1, 2, ..., n\} \) such that no two elements are adjacent (Subsets with 0 or 1 element are convenient). In fact, according
to [8d] (p. 21), the number $F_{n+1}$ of such subsets equals $\sum_{p} \binom{n+1}{p}$.

Hence, it follows that $F_{n+1} = \sum_{p} \binom{n-p}{p-1} + \sum_{p} \binom{n}{p} = F_{n-1} + F_{n}$ (by [8e], p. 10) and $F_0 = F_1 = 1$. Thus, the sequences $F_n$ and $F_{n+1}$ coincide, because they satisfy the same defining recurrence relation. (See also Exercise 13, p. 76, and Exercise 31, p. 86.) It can also be shown that the number $G_n$ of subsets of $[n]$ (p. 24) such that any two points are not adjacent, equals $F_n + F_{n-2}$ (subset $\phi$ is convenient), in other words $G_n = a^n + b^n$, $G_n = G_{n-1} + G_{n-2}$ and $\sum_{n \geq 0} G_n t^n = (1 + t^2)^{-1} (1 - t^2)^{-1}$.

More generally, defining $(1 - t - t^{i+1} t^n)^{-1} = \sum_{n \geq 0} F(n, l) t^n$, it can be proved that $F(n+1, l)$ is the number of subsets $B \subset [n]$ such that any two elements of $B$ are always separated by at least $l(\geq 0)$ elements of $\ominus B$. For subsets $B \subset [n]$ with the same property, the number is $G(n, l)$ where $(l + (l+1) t^{i+1} (1 - t - t^{i+1})^{-1} = \sum_{n \geq 0} G(n, l) t^n$.

(II) Multiple sequences

The concept of GF can be immediately generalized to multiple sequences. We explain the case of double sequences. The three most used GF are the following formal series:

\[ \Phi(t, u) := \sum_{n, k \geq 0} a_{n,k} t^n u^k, \quad \Psi(t, u) := \sum_{n, k \geq 0} a_{n,k} \frac{t^n u^k}{n! k!}, \]

\[ \Theta(t, u) := \sum_{n, k \geq 0} a_{n,k} \frac{t^n u^k}{n!} , \]

the last one, $\Theta$, being especially used in the case of a triangular sequence ($a_{n,k} = 0$, if not $0 \leq k < n$). We now investigate the double sequence of binomial coefficients, $a_{n,k} = \binom{n}{k}$, as an example:

\[ \Phi(t, u) := \sum_{n, k \geq 0} \binom{n}{k} t^n u^k = \sum_{n \geq 0} \left( \sum_{0 \leq k \leq n} \binom{n}{k} u^k \right) t^n = \sum_{n \geq 0} t^n (1 + u)^n = (1 - t (1 + u))^{-1} , \]

which converges if $|t(1 + u)| < 1$.

\[ \Theta(t, u) = \sum_{n, k \geq 0} \binom{n}{k} t^n u^k = \sum_{n \geq 0} \frac{t^n}{n!} (1 + u)^n = \exp\{t(1 + u)\} . \]

\[ \Psi(t, u) = \sum_{n, k \geq 0} \binom{n}{k} \frac{t^n}{n! k!} = \sum_{n \geq 0} \frac{(ut)^k}{k!} \frac{t^n}{n!} (n - k)! \]

\[ = \sum_{k \geq 0} \frac{(ut)^k}{k!} \left[ \sum_{n \geq 0} \frac{t^n}{n!} \right] - (\exp t) \cdot I_0(2\sqrt{ut}) , \]

where $I_0(z) = \sum_{k \geq 0} (z/2)^k k! (k!)^{-1}$ is the modified Bessel function of order 0, because this function is complicated, $\Psi(t, u)$ is not considered very interesting.

(III) General remarks on generating functions

We return to the case of a simple sequence $a_n$.

(1) If the power series $f(z) = \sum_{n \geq 0} a_n z^n$ converges for all complex $z$ ($\iff f(z)$ is an entire function), then the Cauchy integral theorem gives:

\[ a_n = \frac{1}{2\pi i} \int f(z) z^{-n-1} dz , \]

where the integral is taken over a simple curve enclosing the origin, and oriented counterclockwise. Usually, when $f(z)$ is 'elementary', [13] can very well be used for estimating $a_n$ for great $n$ by the Laplace method or the saddlepoint method (see, for instance, [*De Bruijn, 1961*]). In the case that the radius of convergence of $f(z)$ is finite, a Darboux type method can be used (see p. 277).

(2) Of course one can associate with the sequence still others than those of [13a]. For example:

\[ \Omega(t) = \sum_{n \geq 0} a_n \frac{n!}{t^{n+1}} , \]

\[ A(t) = \sum_{n \geq 1} a_n \frac{t^n}{1-t^n} , \]

\[ N(t) = \sum_{n \geq 0} a_n \frac{t^n}{n!} , \]

which are called respectively 'factorial GF' (mostly studied by [*Nörlund, 1924*]).
1.14. List of the principal generating functions

(1) Bernoulli and Euler numbers and polynomials

Bernoulli numbers \( B_n \), Euler numbers \( E_n \), Bernoulli polynomials \( B_n(x) \) and Euler polynomials \( E_n(x) \) are defined by:

\[
\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \quad \frac{t e^t}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}
\]

\[
\frac{2t}{e^t + 1} = \frac{1}{\cosh t} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad \frac{2t e^t}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.
\]

(Many generalizations have been suggested). Bernoulli numbers, denoted by \( b_n \) in Bourbaki, are sometimes also defined by:

\[
t (e^t - 1)^{-1} = 1 - \frac{1}{2} t + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{B_{2k} t^{2k}}{(2k)!}
\]

Each \( B_k \) is then > 0, and equals \((-1)^{k+1} B_{2k} \) as a function of our Bernoulli numbers.

Their most important properties are:

\[
B_n = B_n(0), \quad E_n = 2^n E_n(1)
\]

\[
B_{2k+1} = E_{2k-1} = 0, \quad \text{for} \quad k = 1, 2, 3, \ldots
\]

\[
B'_n(x) = n B_{n-1}(x), \quad E'_n(x) = n E_{n-1}(x)
\]

\[
B_n(x + 1) - B_n(x) = n x^{n-1}, \quad E_n(x + 1) + E_n(x) = 2 x^n
\]

\[
B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k},
\]

\[
E_n(x) = \sum_{k=0}^{n} \binom{n}{k} E_k \left( \frac{x}{2} - \frac{1}{2} \right)^{n-k}
\]

\[
[14h] \quad B_n(1 - x) = (-1)^n B_n(x), \quad E_n(1 - x) = (-1)^n E_n(x).
\]

For instance, \([14d] \) follows from the fact that the functions \( t (e^t - 1)^{-1} - B_0 - B_1 t \) and \( t e^t \) are even; \([14e] \) follows from the fact that, for \( \phi := t e^t - (e^t - 1)^{-1} \), we have \( \phi' = t \phi \), etc. (For a table of \( B_n \) and \( E_n \), see \[*Abramovitz, Stegun, 19641, p. 810, and [Knuth, Buckholtz, 1967] for \( n < 250 \) and \( n < 120 \). Applications are found in Exercises 36 and 37, pp. 88 and 89.) The first values of \( B_n \) and \( E_n \) are:

\[
\begin{array}{cccccccccc}
\hline
n & 0 & 1 & 2 & 4 & 6 & 8 & 10 & 12 \\
B_n & 1 & -1 & 1 & -1 & 5 & -61 & 1385 & -50521 & 2702765 \\
E_n & 1 & 0 & -1 & 5 & -61 & 1385 & -50521 & 2702765 & 1297389
\end{array}
\]

\[
[14h] \quad B_n(1 - x) = (-1)^n B_n(x), \quad E_n(1 - x) = (-1)^n E_n(x).
\]

For instance, \([14d] \) follows from the fact that the functions \( t (e^t - 1)^{-1} - B_0 - B_1 t \) and \( t e^t \) are even; \([14e] \) follows from the fact that, for \( \phi := t e^t - (e^t - 1)^{-1} \), we have \( \phi' = t \phi \), etc. (For a table of \( B_n \) and \( E_n \), see \[*Abramovitz, Stegun, 19641, p. 810, and [Knuth, Buckholtz, 1967] for \( n < 250 \) and \( n < 120 \). Applications are found in Exercises 36 and 37, pp. 88 and 89.) The first values of \( B_n \) and \( E_n \) are:

\[
\begin{array}{ccccccccccc}
\hline
n & 0 & 1 & 2 & 4 & 6 & 8 & 10 & 12 \\
B_n & 1 & -1 & 1 & -1 & 5 & -61 & 1385 & -50521 & 2702765 \\
E_n & 1 & 0 & -1 & 5 & -61 & 1385 & -50521 & 2702765 & 1297389
\end{array}
\]

(For more information about this subject, see, for instance, \[*Campbell, 19661, \[*Jordan, 19471, \[*Nielsen, 19061.\)

We may also define Genocchi numbers \( G_n \) by:

\[
\frac{2t}{e^t + 1} = t (1 + \frac{1}{2} t) = \sum_{n=1}^{\infty} G_n \frac{t^n}{n!}.
\]

Then we have \( G_3 = G_4 = G_5 = \cdots = 0 \) and \( G_{2m} = 2 (1 - 2^m) B_{2m} + 2 m E_{2m-1}(0) \), which shows their close relationship with the Bernoulli numbers (used in Exercise 36, p. 89 for 'computing' \( B_n \)).

\[
\begin{array}{ccccccccccc}
\hline
n & 0 & 1 & 2 & 4 & 6 & 8 & 10 & 12 \\
G_n & 1 & -1 & 1 & -1 & 5 & -61 & 1385 & -50521 & 2702765 \\
\end{array}
\]

(II) Some sequences of 'orthogonal' polynomials

(Their most complete study is made by \[*Szegö, 19671.\)

We list their GF:

\[
[14i] \quad \text{The Chebyshev polynomials of the first kind } T_n(x):
\]

\[
1 - x \frac{1}{1 - 2tx + t^2} = \sum_{n=0}^{\infty} T_n(x) t^n.
\]
The Chebyshev polynomials of the second kind $U_n(x)$:
\[ \frac{1}{1 - 2tx + t^2} := \sum_{n \geq 0} U_n(x) t^n. \]

After some manipulations this implies:
\[ \cos n\varphi = T_n(\cos \varphi), \quad \frac{\sin (n + 1)\varphi}{\sin \varphi} = U_n(\cos \varphi). \]

The Legendre polynomials $P_n(x)$:
\[ \frac{1}{\sqrt{1 - 2tx + t^2}} := \sum_{n \geq 0} P_n(x) t^n. \]

The Gegenbauer polynomials $C^{(\alpha)}(x)$:
\[ (1 - 2tx + t^2)^{-\alpha} := \sum_{n \geq 0} C_n^{(\alpha)}(x) t^n, \]
where $\alpha \in \mathbb{C}$; hence $C_n^{(1/2)} = P_n$, $C_n^{(1)} = U_n$. (These are also called ultraspherical polynomials. See Exercise 35, p. 87.)

The Hermite polynomials $H_n(x)$:
\[ \exp(-t^2 + 2tx) := \sum_{n \geq 0} H_n(x) \frac{t^n}{n!}. \]

The Laguerre polynomials $L_n^{(\alpha)}(x)$:
\[ (1 - t)^{-1-\alpha} \exp \frac{tx}{t-1} := \sum_{n \geq 0} L_n^{(\alpha)}(x) \frac{t^n}{n!} (\alpha \in \mathbb{C}). \]

(III) Stirling numbers

The Stirling numbers of the first kind $s(n, k)$ and of the second kind $S(n, k)$ can be defined by the following double GF:
\[ (1 + t)^n := \sum_{k \leq n} s(n, k) \frac{t^k}{k!} u^k. \]
\[ \exp \{u(e^t - 1)\} := \sum_{k \geq 0} S(n, k) \frac{t^k}{k!} u^k. \]

Because these numbers are very important in combinatorial analysis, we will make a special study of them in Chapter V.

The double GF in their definition can be avoided, if we observe that:
\[ (1 + t)^n = \exp \{u \log (1 + t)\} = \sum_{k \geq 0} \frac{\log^k (1 + t)}{k!} \]
\[ \frac{\log^k (1 + t)}{k!} := \sum_{n \geq k} s(n, k) \frac{t^n}{n!} \]
\[ \exp \{u (e^t - 1)\} = \sum_{k \geq 0} \frac{u^k (e^t - 1)^k}{k!} \]
\[ \frac{(e^t - 1)^k}{k!} := \sum_{n \geq k} S(n, k) \frac{t^n}{n!}. \]

(IV) Eulerian numbers

The Eulerian numbers $A(n, k)$ (not to be confused with Euler numbers $E_n$, p. 48) are generated as follows:
\[ \mathfrak{A}(t, u) := \frac{1 - u}{e^{(u-1)t} - u} := 1 + \sum_{1 \leq k \leq n} \frac{A(n, k) t^n}{n!} u^{k-1}. \]

It is easily verified that:
\[ (u - u^2) \frac{\partial \mathfrak{A}}{\partial u} + (tu - 1) \frac{\partial \mathfrak{A}}{\partial t} + \mathfrak{A} = 0, \]
from which follows, if we put the coefficient of $u^{k-1}t^n/n!$ in this partial differential equation equal to 0, the following recurrence relation:
\[ A(n + 1, k) = (n - k + 2)A(n, k - 1) + kA(n, k), \]
with initial conditions: $A(n, 1) = 1$ for $n \geq 0$ and $A(0, k) = 0$ if $k \geq 2$. Another GF, denoted by $\mathfrak{A}_1$, is sometimes easier to handle:
\[ \mathfrak{A}_1(t, u) := \mathfrak{A} \left( tu, \frac{1}{u} \right) = 1 + u \left( \mathfrak{A}(t, u) - 1 \right) = 1 + \sum_{1 \leq k \leq n} \frac{A(n, k) t^n}{n!} u^{k} = \frac{1 - u}{1 - ue^{(1-u)}}. \]
1.15. Bracketing problems

We will treat in some detail these famous examples of the use of GF.

1) Catalan problem

Consider a product \( P \) of \( n \) numbers \( X_1, X_2, \ldots, X_n \) in this order, \( P = X_1 X_2 \cdots X_n \). We want to determine the number of different ways of putting brackets in this product, each way corresponding to a computation of the product by successive multiplications of precisely two numbers each time ([Catalan, 1838]). Thus, \( a_1 = 1 \), \( a_2 = 2 \) and \( a_4 = 5 \), according to the following list of bracketings:

\[
\begin{align*}
(\underbrace{X_1 X_2} (X_3 X_4)), & \quad \{(X_1 X_2) X_3\} X_4, & \quad \{X_1 (X_2 X_3)\} X_4, \\
(\underbrace{X_1 (X_2 X_3)} X_4), & \quad X_1 \{X_2 (X_3 X_4)\}. \\
\end{align*}
\]

We get then for the list in [15a] the following:

\[
\begin{align*}
(\underbrace{X^2 \cdot X^2}), & \quad (X^2 \cdot X) X, & \quad (X \cdot X^2) X, & \quad (X \cdot X^2) X, & \quad (X \cdot X^2) X. \\
\end{align*}
\]

Notations [15a, b] become quickly clumsy and difficult to handle, but we observe that any nonassociative product also can be represented by a bifurcating tree. Figure 9 (corresponding to \( n=4 \)) shows what we mean. The height of the tree is the number of levels above the root \( R \) (it is 2 for the first tree, and 3 for the four others). There are \( n-2 \) nodes, or bifurcations different from \( R \).

We try to find a recurrence relation between the \( a_n \). The last multiplication, which ends the product of all factors \( X_1, X_2, \ldots, X_n \) in this order, operates on a product of the first \( k \) letters and a product of the last \( n-k \) letters, for some \( k \) such that \( 1 \leq k \leq n-1 \). The first \( k \) letters can be bracketed in \( a_k \) different ways, and the \( n-k \) last ones can be bracketed in different ways. Thus we get, collecting all possibilities as \( k \) ranges over \( [n-1] \):

\[
a_n = \sum_{1 \leq k \leq n-1} a_k a_{n-k}, \quad n \geq 2.
\]

We put:

\[
a_0 := 0, \quad a_1 := 1.
\]

Let now \( \mathfrak{A}(t) \) be the GF of the \( a_n \). Then we get, using [15c] for equality (*) and [15d] for (**) and Theorem B of p. 39 for (***):

\[
\mathfrak{A} = \mathfrak{A}(t) := \sum_{n \geq 0} a_n t^n = \frac{1}{1 - t - \sum_{n \geq 1} a_n t^n}.
\]

In the implication (****), we have considered \( \mathfrak{A} \) as a function of \( t \), hence as solution of the preceding quadratic equation. The expansion of the root with [12e] (p. 37) gives us then the required value of \( a_n \), which is often called the Catalan number:

\[
a_n = \frac{1}{n} \left( \frac{2n-2}{n-1} \right).
\]

We list the first few values of \( a_n \):

\[
\begin{array}{cccccccccccccccccccc}
 n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 \\
 a_n & 1 & 1 & 2 & 5 & 14 & 42 & 132 & 429 & 1430 & 4862 & 16796 & 58786 & 208012 & 742900 & 2674440 & 9694845 & 35357670
\end{array}
\]
Let us finally mention two other representations of Catalan bracketings.

(1) **Triangulations of a convex polygon** (see also Exercise 8, p. 74). The following example clearly explains the rule:

![Triangulations of a convex polygon](image)

(2) **Majority paths** (from André, p. 22). Every path joins A (0, 2) to B (n – 2, n) with the following convention: any opening bracket ( signifies a vertical step and any letter different from X₁ and X₂ a horizontal step.

Using Theorem B (p. 21), with p = n – 2, q = n, we easily obtain [15e].

(II) **Wedderburn-Etherington commutative bracketing problem** ([Wedderburn, 1922], [Etherington, 1937], [Harary, Prins, 1959]. For another aspect of this problem, see [Melzak, 1968].)

We suppose E this time to be **commutative**, and we call the number of interpretations of Xⁿ in the sense of [15b] (p. 52) now bₙ. Thus b₂ = 1 and b₃ = 1, because X² = X.X², b₄ = 2, because (X² · X) X = (X · X²) X = X(X.X²) = X(X² · X). If one prefers, one can also consider bₙ as the number of **binary trees**, two trees being considered identical if and only if one can be transformed into the other by reflections with respect to the vertical axes through the nodes. Thus, Figure 10 shows that b₃ = 3:

![Binary trees](image)

Fig. 10.

We obtain again a recurrence relation, this time again by inspecting the last multiplication performed, but now it depends on whether n is odd or even:

\[
b_{2p-1} = b₁b₂p - 2 + b₁b₂p - 2 + \cdots + b₁b₁p, \quad p > 2.
\]

\[
b_{2p} = b₁b₂p-1 + b₁b₂p-2 + \cdots + b₁b₁p + b₁b₁p + \frac{b₁b₁p + 1}{2}, \quad p > 1.
\]

This can also be written, when we put b₀ := 0, b₁ := 1, b₂ := 1, b₃ := 0 for x ∈ N, as follows:

\[
bₙ = \sum_{0 \leq i < j \leq n} b_i b_j + \frac{1}{2} b_{n/2} + \frac{1}{2} (b_{n/2})^2, \quad n \geq 2.
\]

\[
\mathcal{B}(t) := \sum_{n \geq 0} bₙ tⁿ = t + \sum_{n \geq 2} tⁿ \left( \sum_{\substack{0 \leq i < j \leq n \atop i+j=n}} b_i b_j \right) + \frac{1}{2} \sum_{n \geq 2} b_{n/2} tⁿ + \frac{1}{2} (b_{n/2})^2 tⁿ.
\]

Now:

\[
(1) = \sum_{i \geq 0} b_i b_j tⁿ+j = \frac{1}{2} \left( \sum_{i,j \geq 0} b_i b_j tⁿ+j - \sum_{i \geq 0} b_i² t²i \right)
\]

\[
\frac{1}{2} (\mathcal{B}²(t) - \sum_{i \geq 0} b_i² t²i).
\]

Hence:

\[
\mathcal{B}(t) = t + \frac{1}{2} \mathcal{B}²(t) + \frac{1}{2} \mathcal{B}²(t).
\]

This is a functional equation, which can be simplified by putting \( \mathcal{B}(t) = 1 - \mathcal{B}(t) = 1 - \sum_{n \geq 1} bₙ tⁿ \); then we get:

\[\textbf{15f} \quad \mathcal{B}(t²) = 2t + \mathcal{B}²(t).\]

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<td>36264199</td>
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For a method giving an asymptotic equivalent, see [Otter, 1948]; after a computation due to Bender, \( bₙ \sim 0.3187/662\cdots(2.48325354\cdots/n^{3/2}). \)
We return to the noncommutative case, and we compute the number $c_n$ of bracketings of $X_1, X_2, \ldots, X_n$ where we allow this time in each bracket an arbitrary number of adjacent factors. For example, for $n=4$, we must extend the list of [15a] by the following of Figure 11: (thus $c_4=11$)

$$X_1X_2X_3X_4, \quad X_1(X_2X_3)X_4, \quad X_1X_2X_3X_4, \quad X_1X_2X_3X_4.$$  

Fig. 11.

For a recurrence relation we consider again the last multiplication: this time there are not just two factors to be multiplied, but $l(\geq 2)$, of which $l_1$ factors consist of one letter, $l_2$ of two letters, etc. Hence:

$$[15g] \quad l_1 + l_2 + \cdots + n_{n-1} + n = l,$$

$$l_1 + 2l_2 + \cdots + (n-1) l_{n-1} + nl_n = n,$$

with $l_n=0$, because $l \geq 2$. Now, there are $l!/(l_1!l_2!\ldots l_l!)$ different ways to arrange these $l$ factors of the last operation, because the choice of a particular sequence of these $l$ factors just means giving a $(l_1, l_2, \ldots, l_l)$-division of $[l]$ (cf. p. 27). Hence:

$$c_n = \sum_{l_1 \geq 0} \frac{l!}{l_1!l_2!\ldots l_l!} c_1^{l_1} c_2^{l_2} \cdots c_n^{l_n}, \quad n \geq 2; \quad c_0 := 0, \quad c_1 := 1,$$

where the summation takes place over the $l_1, l_2, \ldots$ such that [15g] and $l \geq 2$ ($\Rightarrow n \geq 2$). Thus:

$$\mathcal{C}(t) := \sum_{n \geq 0} c_n t^n = t + \sum_{n \geq 2} c_n t^n$$

$$= t + \sum_{l_1 \geq 0} \left( \sum_{l_2 \geq 0} \frac{l!}{l_1!l_2!\ldots l_l!} (c_1 t)^{l_1} (c_2 t^2)^{l_2} \ldots \right).$$

$$= t + \sum_{l \geq 2} \left( \sum_{l_1 \geq 0, l_2 \geq 0, \ldots} \frac{l!}{l_1!l_2!\ldots l_l!} (c_1 t)^{l_1} (c_2 t^2)^{l_2} \ldots \right)$$

$$= t + \sum_{l \geq 2} \left( c_1 t + c_2 t^2 + \cdots \right)^l = t + \sum_{l \geq 2} (c_1 t + c_2 t^2 + \cdots)^l = t + \sum_{l \geq 2} (\mathcal{C}^2)^l = t + \frac{\mathcal{C}^2}{1-\mathcal{C}}$$

$$= 2\mathcal{C} - (1 + t) \mathcal{C} + t = 0, \quad \mathcal{C}(0) = 0.$$
make a rectangular lattice consisting of \( n \) vertical lines \( V_i \), each corresponding to an \( x_i \in N \), \( i \in [n] \) and \( n \) horizontal lines \( H_j \), each corresponding as well to an \( x_j \in N \) (in Figure 12, \( n = 7 \)). The points of the intersections of \( V_i \) and \( H_j \) represent the points of \( N^2 \), and each point of \( \mathcal{R} \) is indicated by a little dot \( \bullet \). For instance, in Figure 12, \( x_2 \neq x_5 \), \( x \neq x \).

The points \((x_i, x_j)\), \( i \in [n] \) are the points on the diagonal \( \Delta \) (see p. 3). The lattice representation thus introduced can also be applied to any relation between two sets \( N_1 \) and \( N_2 \), if we think of \( N_1 \) as the 'abscissa', and of \( N_2 \) as the 'ordinate'.

Another representation, called matrix representation of \( \mathcal{R} \subset N_1 \times N_2 \), \( |N_1| = n_1, |N_2| = n_2 \), consists of associating with this relation an \( n_1 \times n_2 \) matrix of 0 and 1, defined by \( a_{ij} = 1 \) if \((x_i, x_j) \in \mathcal{R} \) and 0 otherwise, called the incidence matrix of \( \mathcal{R} \).

**Fig. 12.**

**Definition B.** Let \( \mathcal{R} \) be a binary relation on \( N \), \( \mathcal{R} \subset N^2 \). (I) The reciprocal or inverse relation of \( \mathcal{R} \), denoted \( \mathcal{R}^{-1} \) is defined by \( x \mathcal{R}^{-1} y \iff y \mathcal{R} x \) (the lattice image of \( \mathcal{R}^{-1} \) is hence obtained from the lattice image of \( \mathcal{R} \), by reflection with respect to the diagonal \( \Delta \)). (II) \( \mathcal{R} \) is called total or complete, if and only if for all \((x, y) \in N^2\) \( x \mathcal{R} y \) or \( y \mathcal{R} x \), then the set is called totally ordered. The section \( \langle x \mid \mathcal{R} = \{ y \mid x \mathcal{R} y \} \) is the set of upper bounds of \( x \) and the section \( \langle y \mid \mathcal{R} \rangle = \{ u \mid u \mathcal{R} y \} \) is the set of lower bounds of \( y \). For \( x, y \in N \) the segment \([x, y] \) is the set of \( z \in N \) such that \( x < z < y \).\( x < y \) means \( x < y \) and \( x \neq y \).

A lattice is an ordered set \( N \) such that for each pair \((x, y)\) of elements of \( N \) there exist: (1) an element \( b \in N \), often denoted by \( x \lor y \), which is the smallest element of the set of upper bounds for both \( x \) and \( y \) (also called least upper bound), in the sense that \( x < b, y < b \) and \( x < v, y < v \Rightarrow b < v \); (2) an element \( a \in N \), often denoted by \( x \land y \), the largest lower bound of both \( x \) and \( y \) (also called greatest lower bound), in the sense that \( a \leq x, a \leq y \) and \( u \leq x, u \leq y \Rightarrow u \leq a \).

Finally, we recall the two most important binary relations.

**Definition C.** An equivalence relation \( \mathcal{R} \) on \( N \) is a binary relation, that is reflexive, symmetric and transitive. Then we say that \( x \) and \( y \) are equivalent, if and only if \( x \mathcal{R} y \). The section \( \langle x \mid \mathcal{R} \rangle = \{ y \mid x \mathcal{R} y \} \) is called equivalence class of \( x \): this is the set of \( y \) that are equivalent to \( x \).

The number \( \omega(n) \) of equivalence relations on \( N, |N| = n \), in other words, the number of partitions of \( N \) will be extensively studied (see p. 204).

**Definition D.** An order relation \( \mathcal{R} \) on \( N \) is a binary relation on \( N \), which is reflexive, antisymmetric, and transitive. Often \( x \leq y \) is written instead of \( x \mathcal{R} y \). A set is said to be ordered, if and only if \( x \mathcal{R} y \). The section \( \langle x \mid \mathcal{R} \rangle = \{ y \mid x \mathcal{R} y \} \) is the set of upper bounds of \( x \) and the section \( \langle y \mid \mathcal{R} \rangle = \{ u \mid u \mathcal{R} y \} \) is the set of lower bounds of \( y \). For \( x, y \in N \) the segment \([x, y] \) is the set of \( z \in N \) such that \( x < z < y \).\( x < y \) means \( x < y \) and \( x \neq y \).

A chain with \( k \) vertices (and length \( k - 1 \)) connecting \( x, y \in N \) is a finite set \( z_1, z_2, \ldots, z_k \) such that \( x = z_1 < z_2 < \cdots < z_k = y \). A lattice is an ordered set \( N \) such that for each pair \((x, y)\) of elements of \( N \) there exist: (1) an element \( b \in N \), often denoted by \( x \lor y \), which is the smallest element of the set of upper bounds for both \( x \) and \( y \) (also called least upper bound), in the sense that \( x < b, y < b \) and \( x < v, y < v \Rightarrow b < v \); (2) an element \( a \in N \), often denoted by \( x \land y \), the largest lower bound of both \( x \) and \( y \) (also called greatest lower bound), in the sense that \( a \leq x, a \leq y \) and \( u \leq x, u \leq y \Rightarrow u \leq a \).

The number \( d_0 \) of the order relations on \( N, |N| = n \), equals the number of \( T_0 \)-topologies of \( N \) ([Birkhoff, 1967, p. 117]) and the existence of a simple explicit formula seems completely impossible; even asymptotic
estimates for $d_n$ when $n \to \infty$ turns out to be a very difficult combinatorial problem ([Comtet, 1966], [Harary, 1967], [Kleitman, Rothschild, 1970], [Wright, 1972]. See also Exercise 25, p. 229).

The following is the list of known values of $d_n$ and the numbers $d_n^*$ of the nonisomorphic order relations (two relations are called isomorphic if one can be changed into the other by simply rearranging the numbering of the elements of $N$. The value $d_n^*$ due to [Erné, 1974]).

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
\hline
d_n & 1 & 3 & 19 & 219 & 4231 & 130023 & 6129859 & 431723379 & 44511042511 \\
d_n^* & 1 & 2 & 5 & 16 & 63 & 318 & 2045 & & \\
\hline
\end{array}
$$

Actually, we can introduce the numbers $D(n, k)$ of (labelled) order relations of which the longest chain has $k$ vertices (of course, $d_n = \sum_k D(n, k)$):

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
n/k & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
\hline
1 & 1 & & & & & & & \\
2 & 1 & 2 & & & & & & \\
3 & 1 & 12 & 6 & & & & & \\
4 & 1 & 86 & 108 & 24 & & & & \\
5 & 1 & 840 & 2310 & 960 & 120 & & & \\
6 & 1 & 11642 & 65700 & 42960 & 9000 & 720 & & \\
7 & 1 & 227892 & 2583126 & 2510760 & 712320 & 90720 & 5040 & \\
8 & 1 & 6285806 & 142259628 & 199424904 & 71243760 & 11481120 & 987840 & 40320 \\
\hline
\end{array}
$$

1.17. Graphs

Though we do not want to study graphs, we will sometimes use a little of the language of graph theory, hence this and the next section. We have to make a choice among the various current names of certain concepts, since in this field, the terminology is not yet completely standardized. Actually, this situation has some advantages, as it compels each publication on this subject to define its terms carefully. Any book on graphs can be used as a first introduction to graph theory. (For example [*Berge, 1958], [*Busacker, Saaty, 1965], [*Fiedler, 1964], [*Flament, 1965], [*Ford, Fulkerson, 1967], [Harary, 1967a, b], [Norman, Cartwright, 1965], [Kaufman, 1968a], [König, 1936], [Moon, 1968], [Ore, 1962, 1963, 1967], [Pellet, 1968], [Ringel, 1959],

[*Sainte-Lagué, 1926], [*Sheshu, Reed, 1961], [*Tutte, 1966], and particularly, in the viewpoint adopted here, the attractive book by [Harary, 1969].)

Let $N$ be a finite set. We recall that a pair $B$ of $N$ is a 2-combination, or subset of two elements, p. 7; $B \in \mathcal{P}_2(N)$.

**Definition A.** A graph (over $N$) is a pair $(N, \mathcal{G})$, in which $\mathcal{G}$ is a set (possibly empty) of pairs of $N, \mathcal{G} \subseteq \mathcal{P}_2(N)$. The elements of $N$ are called the nodes or vertices of the graph, and the pairs $(\in \mathcal{G})$ are called edges of the graph. One often says "the graph $\mathcal{G}$" rather than "the graph $(N, \mathcal{G})"$

when the set $N$ is given once and for all.

**Theorem A.** Giving a graph $\mathcal{G}$ on $N$ is equivalent to giving a binary relation $\mathcal{R}$ on $N, \mathcal{R} \subseteq N^2$, which is symmetric and antireflexive, called incidence relation associated with $\mathcal{G}$.

**Define $\mathcal{R}$ by** $x \mathcal{R} y \iff \{x, y\} \in \mathcal{G}$ **■**

A convenient plane representation of a graph consists in drawing the nodes as points and the edges as straight or curved segments, and ignoring their intersections. Figure 13 represents $N := \{a, b, c, \ldots, k, l\}$ and $\mathcal{G} := \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, e\}, \{d, g\}, \{e, k\}, \{f, j\}, \{f, g\}, \{f, j\}, \{h, l\}\}$. 

![Graph Example](image)

**Fig. 13.**

**Definition B.** Let $\mathcal{G}(\in \mathcal{P}_2(N))$ be a graph over $N$. (1) An edge containing a node $x(\in N)$ is called incident with $x$, and $\mathcal{G}(x)$ designates the set of these edges. The number $|\mathcal{G}(x)|$ of edges incident with $x$, also denoted by $\delta(x)$, is called the degree of $x$. Two nodes $x$ and $y$ are called adjacent, if $\{x, y\} \in \mathcal{G}$. Similarly, two edges are called adjacent if they have a node in common. A node is called an end point or terminal node, if its degree
equals 1; the edge adjacent to \( x \) (which is unique) is also called terminal. An isolated node is one with degree 0. (II) \( (N', \mathcal{G}') \) is a subgraph of \( (N, \mathcal{G}) \) if \( N' \subseteq N, \mathcal{G}' \subseteq \mathcal{G}, \mathcal{G}' \subseteq \mathcal{P}_2(N') \); it is called a complete subgraph (or a clique), with support \( N' \) if \( \mathcal{G}' = \mathcal{P}_2(N') \). An independent set \( L(\subseteq N) \) in a graph \( \mathcal{G} \) is a set such that \( \mathcal{P}_2(L) \cap \mathcal{G} = \emptyset \); hence is a complete subgraph of the complementary graph, which is the graph \( \mathcal{G}: = \mathcal{P}_2(N) - \mathcal{G} \). (III) A path or chain connecting \( a \) and \( b (\in N) \) is a sequence of adjacent edges \( \{a_1, x_1\}, \{x_1, x_2\}, \ldots, \{x_{n-1}, b\} \); this path \( \{a, x_1, x_2, \ldots, x_{n-1}, b\} \) is said to have length \( n \) (multiple points may occur, as in the case of the path \( \{j, f, c, d, e, f, g\} \) of Figure 13). A cycle or circuit is a closed path. (For instance, \( \{c, f, g, d, c\} \) in Figure 13.) An Euler circuit is a circuit in which all edges of \( \mathcal{G} \) occur precisely once. A Hamiltonian circuit is a circuit that passes exactly once through every node. (IV) A graph is called connected if every two nodes are connected by at least one path. (V) A tree is a connected acyclic (= without cycles) graph. The distance between two points in a tree is the number of the edges in the (unique) path joining \( a \) with \( b \) (no repetitions of edges allowed to occur in this path).

We indicate now a way to draw a tree \( \mathcal{T} \) of \( N \). We choose a node \( x_0 (\in N) \). From \( x_0 \), we trace the edges connecting \( x_0 \) with the adjacent nodes (those who have distance 1 to \( x_0 \), say \( x_1, x_2, \ldots \)). We arrange these on a horizontal line (Figure 14). From these points, we trace the edges that connect them with the points situated at distance 2 from \( x_0 \) (hence adjacent to \( x_1 \) and not equal to \( x_0 \)), etc. A tree in which such a special point \( x_0 \), the root, has been chosen, is also called rooted tree. The preceding construction proves Figure 14.

**Theorem B.** Each tree has at least two endpoints, and for \( n \geq 3 \), at least two terminal edges.

Another characterization of trees is:

**Theorem C.** Any two of the following three conditions (1), (2) and (3) imply the third, and moreover, imply that the graph \( \mathcal{G} \) over \( N, |N| = n \) is a tree: (1) \( \mathcal{G} \) is connected; (2) \( \mathcal{G} \) is acyclic; (3) \( \mathcal{G} \) has \( (n-1) \) edges.

\[ (1), (2) \Rightarrow (3). \] In other words, by Definition B (V), any tree with \( n \) vertices has \( n-1 \) edges. This is true for \( n = 2 \). We prove the statement by complete induction, and we suppose it to be true for all trees having up to \( (n-1) \) edges. In a tree \( \mathcal{G} \) with \( n \) nodes, we cut off one of the terminal nodes and its incident edge. The new graph obtained in this way is evidently a tree, hence it contains \( (n-1) \) nodes, so \( |\mathcal{G}'| = n-2 \) according to the induction hypothesis; hence \( |\mathcal{G}| = n-1 \).

\[ (1), (3) \Rightarrow (2). \] We reason by reductio ad absurdum. Suppose that there exists \( (N, \mathcal{G}), |N| = n, |\mathcal{G}| = n-1 \), which is connected, and with at least one cycle \( \mathcal{C} \). We break the cycle \( \mathcal{C} \) by omitting one edge. Thus we obtain a new graph \( (N, \mathcal{G}_1) \), still connected, with \( |\mathcal{G}_1| = n-2 \). We repeat this operation until there are no cycles left, so we have a connected acyclic graph \( (N, \mathcal{G}_i) \), with \( n-1-i \) edges, for some \( i \geq 1 \), which contradicts the statement that (1), (2) imply (3).

\[ (2), (3) \Rightarrow (1). \] If not, there exists \( (N, \mathcal{G}), |N| = n, |\mathcal{G}| = n-1 \), with two nodes \( a, b \in N \) not connected by a path of \( \mathcal{G} \). If we connect \( a \) and \( b \) by a new edge \( \{a, b\} \), we obtain a new graph \( (N, \mathcal{G}_1) \), which is still acyclic, with \( |\mathcal{G}_1| = n \). Repeating this procedure, we finally obtain a connected acyclic graph \( (N, \mathcal{G}_i) \) with \( n-1+i \) edges, for some \( i \geq 1 \), which again contradicts that (1), (2) imply (3).

Let us now prove the famous Cayley theorem ([Cayley, 1889]).

**Theorem D.** The number of trees over \( N, |N| = n \), equals \( n^{n-2} \).

There are many proofs of this theorem. One kind, of constructive type,
establishes a bijection between the set of trees over \([n]\) and the set \([n]^{(s-2)}\) of \((n-2)\)-tuples of \([n]\), \((x_1, x_2, \ldots, x_{n-2})\), \(x_i \in [n]\). ([Foata, Fuchs, 1970], [Neville, 1953], [Prüfer, 1918]), and, for a generalization to \(k\)-trees, [Foata, 1971]. See also p. 71.) Others follow the path of obtaining the various enumerations suggested by the problem. ([Clarke, 1958], [Dziobek, 1917], [Katz, 1955], [Mallows, Riordan, 1968], [Moon, 1963, 1967a, b], [Riordan, 1957a, 1960, 1965, 1966], [Rényi, 1959, 1959].) We give here the proof of Moon, which is of the second type.

**Theorem E.** Let \(T = T(N; d_1, d_2, \ldots, d_n)\) be the set of trees over \(N := \{x_1, x_2, \ldots, x_n\}\) whose node \(x_i\) has degree \(d_i(\geq 1)\), \(i \in [n]\), where \(d_1 + d_2 + \cdots + d_n = 2(n-1)\). Then:

\[
[17a] \quad T(n; d_1, d_2, \ldots, d_n) := [T(N; d_1, d_2, \ldots, d_n)] = (d_1 - 1, d_2 - 1, \ldots, d_n - 1).
\]

(We use here the notation for the multinomial coefficients introduced in [10c], p. 27.)

It is clear that \(T(n; d_1, d_2, \ldots, d_n) = 0\) if \(d_1 + d_2 + \cdots \neq 2(n-1)\), because every tree over \(N\) has \((n-1)\) edges (Theorem C., p. 63). We first prove three lemmas.

**Lemma A.** Let integers \(b_i \geq 1, i \in [s]\), be given such that \(\sum_{i=1}^{s} b_i = m\). Then:

\[
[17b] \quad (b_1, \ldots, b_s) = \sum_{k=1}^{s} (b_1, b_2, \ldots, b_k - 1, \ldots, b_s).
\]

(So, this formula is a generalization of the binomial relation \((b, c) = (b-1, c) + (b, c-1)\), [5e] p. 10.)

**Lemma B.** Let be given integers \(a_j \geq 0, j \in [t]\) such that \(\sum_{j=1}^{t} a_j = m\). Then:

\[
[17c] \quad (a_1, a_2, \ldots, a_t) = \sum_{j=1}^{t} (a_1, a_2, \ldots, a_j - 1, \ldots, a_t),
\]

where the summation is taken over all \(j\) such that \(a_j \geq 1\). (If not, then the multinomial coefficient under the summation sign equals 0 by definition. Compare with [Tauber, 1963]).

Now we return to [17a], and we suppose that:

\[
[17d] \quad d_1 \geq d_2 \geq \cdots \geq d_n.
\]

This amounts to changing the numbering of the \(x_i\).

**Lemma C.** Summing over the \(i\) such that \(d_i \geq 2\), the following holds,

\[
[17e] \quad T(n; d_1, d_2, \ldots, d_n) = \sum_{i, d_i \geq 2} T(n-1; d_1, \ldots, d_{i-1}, d_i-1, \ldots, d_n-1).
\]

It follows from [17d] and from Theorem B that \(d_n = 1\). Let \(T_i := \{ \mathcal{T} \mid x_i \in \mathcal{T}, \text{x}_i \text{ adjacent to } x_i \}\). Hence \(i \leq n-1\) and \(d_i \geq 2\). Now we have the division \(T = \sum T_i\), where we sum over all \(i\) such that \(d_i \geq 2\). Hence [17e], if we observe that

\[
|T_i| = |T(N - \{x_i\}; d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n)|.
\]

**Proof of Theorem E.** We prove formula [17a] by induction. It is clearly true for \(n = 3\). Suppose true for \(n - 1\) and smaller. Then, with [17e] and the induction hypotheses for equality (*), \(d_n = 1\) for (***) and [17c] for (***):

\[
T(n; d_1, d_2, \ldots, d_n) = \sum_{i, d_i \geq 2} (d_1 - 1, \ldots, d_i - 2, \ldots, d_n - 1) (***) \sum_{i, d_i \geq 2} (d_1 - 1, \ldots, d_i - 2, \ldots, d_n - 1) (***) [17a].
\]

**Theorem F.** The number \(L(n, k)\) of trees \(\mathcal{T}\) over \(N\) such that a given node, say \(x_n\), has degree \(k\), equals:

\[
[17f] \quad L(n, k) = \binom{n-2}{k-1} (n-1)^{n-k-1}.
\]
We have, using \([17a]\) for equality (*) and \(c_i=d_i-1, i\in[n-1]\) for (**), and \([10f]\) (p. 28) for (***):

\[
L(n, k) = \sum_{d_1 + \ldots + d_{n-1} = 2n-2-k} (d_1 - 1, \ldots, d_{n-1} - 1, k-1)
\]

\[= \binom{n-2}{k-1} \sum_{c_1 + \ldots + c_{n-1} = n-k-1} (c_1, c_2, \ldots, c_{n-1}, k-1)
\]

\[= \frac{(n-2)!}{(k-1)!} (n-1)^{n-k-1}.
\]

**Proof of Theorem D.** By Theorem F, the total number of trees over \(N\) equals:

\[
\sum_{k>1} I(n, k) = \sum_{1<\ell<n-1} \binom{n-2}{k-1} (n-1)^{n-k-1}
\]

\[= \left[1 + (n-1)\right]^{n-2} = n^{n-2}.
\]

To finish this section on graphs, we discuss the Hasse diagram of an order relation over \(N\). This graph is obtained by joining \(a\) and \(b\) if and only if \(a \leq b\) and \(a \leq c \leq b\Rightarrow c = a\) or \(c = b\) (\(\Rightarrow b\) covers \(a\)). In this case \(b\) is placed over \(a\). For example, Figure 15 is the Hasse diagram of the order relation \(\leq\) on \(N = \{a, b, c, d, e, f, g, h, i, j\}\) defined by \(a \leq b, a \leq d, b \leq c, d \leq e, d \leq f, e \leq c, f \leq c, g \leq i, g \leq h\). If one wants to avoid, in this diagram, the difficulty of putting every point on different heights, then one must orient the edges; in this case one obtains a transitive digraph, as in Figure 16.

**1.18. Digraphs; Functions From a Finite Set Into Itself**

(1) **Digraphs in General**

We call a 2-arrangement \((x, y)\) of \(N\) an ordered pair, that is a pair in which we distinguish a first element, \((x, y)\in\mathcal{U}_2(N)\), (see p. 6).

**Definition A.** A digraph \((N, \mathcal{D})\) or directed graph (over \(N\)) is a pair, is such that \(D\) is a (possibly empty) set of ordered pairs from \(N, \mathcal{D}\subset\mathcal{U}_2(N)\). The elements of \(N\) are then called the nodes or vertices of the digraph, and the ordered pairs are called the arcs. One often says “digraph \(\mathcal{D}\)”, rather than “digraph \((N, \mathcal{D})\)”, in case the set \(N\) is given once and for all.

Most of the concepts introduced in the previous section have their analogue in digraphs. For instance, the outdegree of \(x\in N\), denoted by \(od(x)\) is the number of arcs leaving \(x\); the indegree, denoted by \(id(x)\) is the number of arcs entering \(x\). An oriented cycle is a cycle on which the orientation of the arcs is such that of two consecutive arcs always the first one is entering their common node, and the other is leaving it (or vice versa). Other definitions are adapted in the same manner.

**Theorem A.** Giving a digraph \(\mathcal{D}\) over \(N\) is equivalent to giving an antireflexive binary relation \(\mathcal{J}\) on \(N\) called the incidence relation of \(\mathcal{D}\).

**Define** \(\mathcal{J}\) by: \(x\mathcal{J}y \iff (x, y)\in\mathcal{D}\).

There is again a plane representation, analogous the one introduced on p. 61, but with arrows added. Figure 17 shows a digraph and its associated relation. If the relation was not antirefexive, we had to introduce loops into the digraph. But digraphs with loops permitted and relations are the same.
(II) Tournaments

**Definition B.** A tournament (over \( N \)) is a digraph \( D \) such that every pair \( \{x_i, x_j\} \in \mathcal{P}_2(N) \) is connected by precisely one arc. If the arc \( x_i x_j \) belongs to \( D \), we say that \( x_i \) dominates \( x_j \). The score \( s_i \) of \( x_i \) is the number of nodes \( x_j \) that are dominated by \( x_i \). Usually, the nodes \( (\in N) \) of \( D \) are numbered in such a way that:

\[ (18a) \quad 0 \leq s_1 \leq s_2 \leq \cdots \leq s_n = (\leq n-1) \]

The \( n \)-tuple \((s_1, s_2, \ldots, s_n)\in\mathbb{N}^n\) is then called the score vector of \( D \).

The relation \( \mathcal{J} \) (the incidence relation on \( N \)) associated with \( D \) is hence total, antireflexive and antisymmetric. Figure 18 represents a tournament in which \( s_1=s_2=1, s_3=s_4=2 \).

**Theorem B.** A sequence \((s_1, s_2, \ldots, s_n)\) of integers such that \[(18a)\] holds, is a score vector if and only if:

\[ (18b) \quad \sum_{i=1}^{n} s_i = \binom{n}{2} \]

\[ (18c) \quad \text{For all } k \in \mathbb{Z}, \sum_{i=1}^{k} s_i \geq \binom{k}{2}. \]

We only show that the condition is necessary. (For sufficiency, see the beautiful book by [*Moon, 1968*] on tournaments, or the papers by [Landau, 1953] or [Ryser, 1964]. The reader is also referred to [*André, 1900*] and [André, 1898–1900].) For all \( x \in N \) let \( \mathcal{A}(x) \) be the set of arcs issuing from \( x \), \( |\mathcal{A}(x)| = s_i \); \[(18b)\] follows then from considering the cardinalities in the division \( \sum_{i=1}^{n} \mathcal{A}(x_i) = D \). On the other hand, for all \( K \subseteq N \), the set of \( \binom{k}{2} \) arcs whose two nodes belong to \( K \), clearly is contained in \( \sum_{x \in K} \mathcal{A}(x) \); hence \[(18c)\], by considering the cardinalities of the sets involved.

(III) Maps of a finite set into itself

**Definition C.** A digraph over \( N \) is called functional if the outdegree of every node equals 0 or 1; \( \forall x \in N, od(x) \leq 1 \).

There exists a bijection between the set \( \mathcal{N}^N \) of maps \( \varphi \) of \( N \) into itself and the set of such digraphs \( D \). In fact, we may associate \( D \) with \( \varphi \) by \((x, y) \in D \Leftrightarrow \varphi(x) = y, y \neq x \). In this case \( D \) is called the 'functional digraph associated with \( \varphi \)'. Figure 19 corresponds to a \( \varphi \in [22]^{22} \).

**Theorem C.** The relation \( \mathcal{E} \) on \( N \) defined by: \( x \mathcal{E} y \Leftrightarrow \exists \varphi \in \mathcal{N} \), \( \exists q \in \mathcal{N} \) such that \( \varphi^q(x) = \varphi^q(y) \) is an equivalence relation. The restriction of \( \varphi \) to each class of \( \mathcal{E} \) has for associated digraph an oriented cycle, to which (possibly) some trees are attached. Such a digraph is sometimes called an 'excycle' (Weaver).

The classes of \( \mathcal{E} \) are the connected components of \( D \). In the case of Figure 19, there are 5 excycles. In this way each map \( \varphi \in \mathcal{N}^N \) can be decomposed into a product of disjoint excycles, this result being analogous to the decomposition of a permutation into cyclic permutations. (For
other properties of \( N^N \) see, for example, [Dénes, 1966, 1968], [Harary, 1959b], [Hedrlik, 1963], [Read, 1961], [Riordan, 1962a], [Schützenberger, 1968]. For the 'probabilistic' aspect see [Katz, 1955], [Purdom, Williams, 1968].

**Definition D.** A map \( \varphi \in N^N \) is called acyclic if each of its excycles is a rooted tree. In other words, giving \( \varphi \) is equivalent to giving a rooted forest over \( N \), i.e. a covering of \( N \) by disjoint rooted trees.

For instance, the map \( \varphi \) of Figure 19 is not acyclic, but the following is:

\[ \psi(i) = i+1 \quad \text{for } i \in \{21\} \quad \text{and} \quad \psi(22) = 22. \]

**Theorem D.** The number of acyclic maps of \( N \) into itself, that is, the number of rooted forests over \( N \), \( |N| = n \), equals \( (n+1)^{n-1} \).

We adjoin a point \( x \) to the set \( N \), and we let \( P := \{ x \} \cup N \); \( |P| = n+1 \).

Each tree \( T \) over \( P \) becomes a rooted forest if we chop off the branches issuing from \( x \). We call this rooted forest \( \varphi(T) \). Its roots are just the nodes adjacent to \( x \) in \( T \). This map establishes, evidently, a bijection between the rooted forests over \( N \) and the trees over \( P \), hence by Theorem D (the Cayley theorem) (p. 63), \( |P|^{n-2} = (n+1)^{n-1} \).

**Theorem E.** The number of acyclic maps of \( N \) into itself, with exactly \( k \) roots, equals \( \binom{n-1}{k-1} n^{n-k} \).

As before, by joining a \( (n+1) \)-th point \( x \) to each root, we get a tree with \( n+1 \) nodes, in which \( x \) has degree \( k \). Then apply \([17f]\) (p. 65).

**IV** Coding functions of a finite set ([Foata, 1970]).

After labeling, we can work with the set \([n]^k := \{1, 2, 3, \ldots, n\} \). Let us explain how to represent any map \( f \in [n]^k \) into itself, that is to say any function \( f : [n]^k \to [n]^k \), by a word \( x \rightarrow x(f) \) in the noncommutative indeterminates (or letters) \( x_1, x_2, \ldots, x_n \), where each \( x_i \) is identified with the element (or label) \( i \in [n] \).

Every cycle of \( f \) (p. 69) supplies letters of a word, whose first letter, or label, is its greatest element, the other letters following in the opposite direction of the arrows. For example, the cycle \((5 \rightarrow 21 \rightarrow 11 \rightarrow 5)\) of Figure 19 gives \( x_2 \times x_3 x_{11} \). Now, juxtaposing left to right the preceding words by increasing letters, we get a word \( w_0 \) which represents the cyclic label \( x(f) \). Here, \( w_0 = x_0 x_1 x_2 x_3 \).

Considering then the first leaf (terminal node \( x \) of the digraph, such that \( ix = 0 \), \( odx = 1 \), with the smallest label), we construct a word \( w_1 \) which is the path joining this leaf to \( w_0 \), leaf excluded, root included, but written from root to leaf. Here the first leaf being \( 3 \), we have \( w_1 = x_2 x_8 \).

The number of acyclic maps of \( N \) into itself, that is, the enumerator of \( N \), \( E = \sum_{f \in [n]^k} \ell(f) \). Let us give a few examples. (1) If \( E_{[n]^k} \), then \( E_{[n]^k} = t_{1} t_{2} + \cdots + t_{n} \). (2) If \( E \) is the set of functions of \([n] \) for which \( 1, 2, 3, \ldots \) is fixed points, then \( E = t_{1} t_{2} + \cdots + t_{n} = 2^{n-1} \). (3) If \( E \) is the set of acyclic functions whose fixed points (roots) are \( 1, 2, 3, \ldots, k \), then \( E = (t_{1} + \cdots + t_{k})^{k-1} \). Of course, \( E_{[n], 1, 1, \ldots} = |E| \).
So, the three preceding examples allow us to obtain (again) the numbers (1) \( n^n \) of functions of \([n]\), (2) \( n^{n-k} \) of functions with \( k \) given fixed points, (3) \( k \cdot n^{n-k-1} \) of trees with \( k \) given roots (especially Cayley if \( k = 1 \)). Similarly, the coefficient of \( t_1 t_2 \ldots \) in \( F_k \) \((t_1, t_2, \ldots)\) is the number of \( f \in E \) such that \( x(f) \) has \( 1 \) occurrences of \( n_1 \), \( 2 \) occurrences of \( n_2 \), etc.

For any division of \( E \), \( E = E_1 + E_2 + \ldots \), we have \( T_k = T_{E_1} + T_{E_2} + \ldots \) obviously. Finally, let us consider a division of \([n]\), \([n] = \sum A_i\), and a family of sets \( E_i \) of functions, \( E_i \subset [n]^{\ast} \), having the following property: every \( f \in E_i \) acts on \( A_i \) only, i.e. \( f \in E_i \iff \forall x \notin A_i \), \( f(x) = x \). Then the set \( E = E_1 E_2 E_3 \ldots \) of all functions which can be factorized \( f = f_1 f_2 f_3 \ldots \) (in the sense of the composition of functions, here commutative), where \( f_1 \in E_1, f_2 \in E_2, \ldots \) is such that \( T_{E_1} = T_{E_2} \cdot T_{E_3} \ldots \).

**Supplement and Exercises**

(As far as possible we follow the order of the sections.)

1. **n points in a plane.** Let \( N \) be a set of \( n \) points or nodes in the plane such that no three among them are collinear. Moreover, we suppose that each pair among the \( \binom{n}{2} \) straight lines connecting each pair of points is intersecting, and also no three among these lines have a point in common other than one of the given nodes. Show that these \( \binom{n}{2} \) lines intersect each other in \( \frac{1}{2} n(n-1)(n-2)(n-3) \) points different from those in \( N \), and that they divide the plane into \( \frac{1}{6} (n-1)(n^2-5n^2+18n-8) \) (connected) regions, including \( n(n-1) \) unbounded regions.

\*2. **Partitions by lines, planes, hyperplanes.** (1) Let be given \( n \) lines in the plane, each two of them having a point in common but no three of them having a point in common. These lines divide the plane into \( \frac{1}{2} (n^2+n+2) \) regions. [Hint: Show that the number \( a_n \) which is asked satisfies the relation \( a_n = a_{n-1} + n \), \( a_1 = 2 \).] (2) More generally, \( n \) hyperplanes in \( R^k \), in general position, determine \( a(n, k) \) 'regions', with \( a(n, k) = \sum_{i=0}^{k} \binom{n}{i} = 2^n - \sum_{i=0}^{k-1} 2^i \binom{n-1}{k} \); the number of bounded regions is \( \binom{n-1}{k} \).

3. **Circles.** \( n \) circles divide the plane into at most \( n^2-n+2 \) regions. The \( \binom{n}{3} \) circles that are the circumscribed circles of all triangles whose vertices lie in a given set \( N \) of \( n \) points (in general position) in the plane, intersect each other in \( \frac{1}{2} (n^3-3n^2+2n-1) \) points different from those of \( N \).

4. **Spheres.** \( n \) spheres divide the 3-dimensional space into at most \( n(n^2-3n+8)/3 \) regions; \( n \) great circles divide the surface of a sphere into at most \( n^2-n+2 \) regions. More generally, \( n \) hyperspheres divide \( R^k \) into at most \( \binom{n-1}{k} + \sum_{i=0}^{n} \binom{n}{i} \) regions.

5. **Convex polyhedra.** \( F, V, E \) stand for the number of faces, vertices and edges of a convex polyhedron. To show the famous Euler formula \( F+V-E=1 \) can be shown to hold by induction on the number of faces (["Grünbaum, 1967"] gives a thorough treatment of polytopes in arbitrary dimension \( d \), with an abundance of bibliography and of open problems. See also ["Klee, 1966"]).

6. **Inscribed and escribed spheres of a tetrahedron.** Let be given a tetrahedron \( T \), and let \( A_1, A_2, A_3, A_4 \) be the areas of its four faces. To show that the number of spheres which are tangent to all four planes that contain the faces of \( T \) (inscribed and escribed) is equal to \( 8-s \), where \( s \) is the number of equalities satisfied by \( A_1, A_2, A_3, A_4 \), the equalities being taken from \( A_1 + A_2 = A_1 + A_3, A_1 + A_3 = A_2 + A_4, A_1 + A_4 = A_2 + A_3 \) (hence \( 0 \leq s \leq 3 \)). If possible, generalize to higher dimensions. (See ["Vaughan, Gabai, 1967"] and ["Gerber, 1972"]).

\*7. **Triangles with integer sides.** (1) The number of non-congruent tri-
angles with integer sides and given perimeter \( n \) equals \([\sqrt[4]{2} (n^2 + 3n + 21 + (-1)^{n-1} 3n)]\) \([\lfloor x \rfloor \) denotes here the largest integer smaller or equal to \( x \), also called the integral part of \( x \). (2) The number of triangles that can be constructed with \( n \) segments of lengths 1, 2, ..., \( n \) equals \( \frac{1}{2} \left( 1 + (-1)^n \right) + \frac{1}{2} \left( \frac{n+2}{2} \right) + \frac{1}{2} \left( \frac{n+3}{3} \right) \).

8. Some enumeration problems related to convex polygons. Let \( A_1, A_2, ..., A_n \) be the \( n \) vertices of a convex polygon \( P \) in the plane. We call diagonal of \( P \), any segment \( A_iA_j \) which is not a side of \( P \). We suppose that any three diagonals have no common point, except a vertex. (1) Show that the diagonals intersect each other in \( \left( \begin{array}{c} n \cr 4 \end{array} \right) \) interior points of the polygon, and in \( \frac{1}{2} n(n-3)(n-4)(n-5) \) exterior points. (2) The sides and the diagonals divide the interior of \( P \) into \( \frac{1}{2} (n-1)(n-2)(n^2-3n+12) \) convex regions (in the case of Figure 20, we have 11 such regions), and the whole plane into \( \frac{1}{2} (n^2-6n^3+23n^2-26n+8) \) regions.

(3) The number \( d_n \) of ways to cut up the polygon \( P \) into \( (n-2) \) triangles by means of \( n-3 \) nonintersecting diagonals (triangulations of \( P \)) equals \( (n-1)^{-1} \left( \begin{array}{c} 2n-4 \cr n-2 \end{array} \right) \), the Catalan number \( d_{n-1} \) of p. 53; so, this number is that of well-bracketed words with \( (n-1) \) letters. (The heavy lines in Figure 20 give an example of such a triangulation.) [Hint: Choose a fixed side, say \( A_1A_2 \); from each triangulation, remove the triangle with \( A_1A_2 \) as side; then two triangulated polygons are left; hence \( d_n = d_3d_{n-1} + d_7d_{n-2} + ... + d_{n-1}d_2 \); then check the formula, or use [15c] of p. 53.]

Moreover, \( 2(n-3) d_n = n(d_3d_{n-1} + d_7d_{n-2} + ... + d_{n-1}d_3) \). [Hint: Use the two triangulated polygons on each side of each of the \( 2(n-3) \) diagonal vectors \( A_iA_j \).] (Guy, 1967a). Very interesting generalizations of the concept of triangulation are found in the papers by Brown, Mullin and Tutte cited in the bibliography.) Finally, there are \( n^{2n-5} \) triangulations in which each triangle has at least one side which is side of \( P \), \( n \geq 4 \).

(4) There are \( \frac{n-3}{d(n-1)} \left( \begin{array}{c} n \cr 2 \end{array} \right) \) ways of decomposing \( P \) into \( d \) subsets with \( a-1 \) diagonals that do not intersect in the interior of the polygon ([Prouhet, 1866]). (5) There are \( \frac{1}{6} (n+1) \left( n^3 + 18n^2 + 43n + 60 \right) \) triangles in the interior of \( P \) such that every side is side or diagonal of \( P \). (6) Suppose \( n \) even. The number of graphs with \( n/2 \) edges that intersect each other outside of the polygon, equals \( (n+1)^{-1} \left( \begin{array}{c} n \cr 1 \end{array} \right) \). (See [Yang, 1961].) (7) The number of broken open lines without self-intersections (the number of piecewise linear homeomorphic images of the segment \([0, 1]\) contained in the union of \( P \) with its diagonals) whose vertices are vertices of \( P \), equals \( n^{2n-3} \). (In Figure 20, BCAED is an example of such a line.) ([Camille) Jordan, 1920].)

9. The total number of arrangements of a set with \( n \) elements. This number \( P_n := \sum_{k=0}^{n} (n)_k \) satisfies \( P_n = nP_{n-1} + 1 \), \( n \geq 1 \), \( P_0 := 1 \) and \( P_n = n! \times \sum_{k=0}^{n} \left( \frac{1}{k!} \right) \). Hence \( P_n \) equals the integer closest to \( e^n \). Moreover, we have an GF: \( \sum_{n=0}^{\infty} P_n z^n / n! = e^z (1 - z)^{-1} \).

10. 'Binomial' expansions of an integer. Let \( k \) be an integer \( \geq 1 \). With every integer \( n \geq 1 \) is associated exactly one sequence of integers \( b_i \) such that \( n = \left( \frac{b_1}{1} + \frac{b_2}{2} + ... + \frac{b_k}{k} \right) \) and \( 0 \leq b_1 < b_2 < ... < b_k \). There also exists \( 0 \leq c_1 < c_2 < ... < c_k \) such that \( n = \left( \frac{c_1+1}{1} + \frac{c_2+2}{2} + ... + \frac{c_k+k}{k} \right) \).
11. **Greatest common divisor of several integers.** Let \( N = \{a_1, a_2, \ldots, a_n\} \) be a set of \( n \) integers \( \geq 1 \). Let \( P_k \) be the product of the \( \left( \begin{array}{c} n \\ k \end{array} \right) \) LCM’s of all the \( k \)-blocks of \( N \); show that the GCD of \( N \) equals \( \prod P_k \).

12. **Partial sums of the binomial expansion.** Show that for \( 0 \leq k \leq n - 1 \):

\[
\sum_{i=0}^{k} \binom{n}{i} a^{n-i} b^i - (n-k) \binom{n}{k} \int_0^1 t^k (a + b - t)^{n-k-1} \, dt
= (n-k) \binom{n}{k} (a+h)^n \int_0^{n/h} \frac{u^{n-k-1}}{(1+u)^{n+k+1}} \, du
\]

(See also Exercise 2, (2), p. 72.)

13. **Transversals of the Pascal triangle.** Show that \( \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots = F_\ast \), the Fibonacci number (see p. 45) and \( F_n = \sum_{k \leq n} \binom{n-k}{k} x^k \) where \( \alpha, \beta = (1 \pm \sqrt{1+4x})/2 \). More generally, let \( a, v, w \) be integers such that \( v \geq 0, w \geq 1, u < w \) and let:

\[
a_s = a_s (u, v, w) = \binom{n}{v} + \binom{n+u}{v+w} + \binom{n+2u}{v+2w} + \cdots ;
\]

then

\[
\sum_{s \geq 0} a_s t^s = \frac{t^u (1-t)^{n-u-1}}{(1-t)^v - t^w-u}.
\]

(["Riordan, 1958", p. 40. See also Exercise 26, p. 84.)

14. **The number of binomial coefficients.** For each set \( E \) of integers \( \geq 0 \),

\[E = \{a_1, a_2, \ldots, a_m\} \subseteq N^+\]

be a set of \( n \) integers \( \geq 1 \). Let \( P_k \) be the product of the \( \binom{n}{k} \) LCM’s of all the \( k \)-blocks of \( N \); show that the GCD of \( N \) equals \( \prod P_k \), where \( k \) and \( n \) are variables with \( 2 \leq k \leq n \).

\[\text{Hint: } E_n = \left\{ \binom{n}{k} \mid n \geq 2k \right\} \text{; then } E = \bigcup_{k \geq 2} E_k \text{; hence: } |E_2(x)| \leq |E(x)| \leq \leq |E_2(x)| + \sum_{k \geq 3} |E_k(x)|.\]

(For a generalization to multinomial coefficients, see [Erdős, Niven, 1954].)

15. **Generalization of \( \binom{n}{k} \equiv 0 \pmod{p} \) to multinomial coefficients ([André, 1873]).** Let \( M = \{a_1, a_2, \ldots, a_m\} \subseteq N^+ \) be a set of integers \( \geq 1 \), not necessarily distinct. For \( n = a_1 + a_2 + \cdots + a_m \) Theorem A (p. 27) shows that the number \( n!/(a_1! a_2! \cdots a_m!) \) is always an integer. This property can be refined as follows. We put, for each integer \( d \geq 2 \), \( M(d) = \{x \mid x \in M, d \text{ divides } x\} \) and we let \( \gamma (M) = \max_{d \geq 2} |M(d)|. \)

\[0 \leq \gamma (M) \leq m, \text{ and } \gamma (M) = m \text{ if the } a_i \text{ are relatively prime, } \gamma (M) = 1 \text{ if each two among the } a_i \text{ are relatively prime, and } \gamma (M) = 0 \text{ if the } a_i \text{ equal 1.}
\]

Show then that the number \( n!/(a_1! a_2! \cdots a_m!) \) is always an integer (for \( n \) prime and \( m = 2 \) we recover Theorem C, p. 14).}

16. **Polynomial coefficients ([André, 1875], [Montel, 1942]).** This is the name we give to the coefficients of \( f(t) = (1+t+t^2+\cdots+t^{q-1})^r = \sum_{k \geq 0} \binom{x}{k} q \binom{k}{x} t^x \), for \( q \) arbitrary integer \( \geq 0 \), and complex \( x, t \). Evidently, \( \binom{x}{k} = \binom{x}{k} \) and \( \left(-x, \infty, \infty\right) = \binom{x}{k} \). Then \( \binom{x}{k} = \sum_{j=k}^{n} \binom{x}{j} \binom{j}{k} \), where \( q_j + j = k. \]

\[\text{Hint: } f = (1-t^q)^r (1-t)^{-q}. \]

(2) If \( x = n \) is an integer \( \geq 0 \), then \( \binom{n}{k} \) is the number of \( k \)-combinations of \( [n] \) having less than \( q \) repetitions. Generalize the most important properties of the \( \binom{n}{k} \) to these combinatorial coefficients: arithmetical triangle, recurrence relations, congruences, etc., and prove the formula

\[\binom{n}{k} = \frac{2}{\pi} \int_{0}^{\pi/2} \frac{\sin q \theta}{\sin \theta} \cos (n(q-1) - 2k) \theta \, d\theta .
\]

Using this integral representation, find the asymptotic equivalent
Here are the first values of trinomial coefficients \( \binom{n}{k} \):

\[
\begin{array}{cccccccccccc}
  n \backslash k & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
  0 & 1 \\
  1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
  2 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
  3 & 1 & 3 & 6 & 7 & 6 & 3 & 1 & 1 & 3 & 6 & 1 \\
  4 & 1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1 \\
  5 & 1 & 5 & 15 & 30 & 45 & 51 & 45 & 30 & 15 & 5 \\
  6 & 1 & 6 & 21 & 50 & 90 & 126 & 126 & 90 & 50 & 21 & 6 \\
  7 & 1 & 7 & 28 & 77 & 161 & 266 & 357 & 357 & 266 & 161 & 77 \\
  8 & 1 & 8 & 36 & 112 & 266 & 504 & 784 & 1016 & 1264 & 1128 & 784 \\
\end{array}
\]

and of quadrinomial coefficients \( \binom{n}{k} \):
19. Sequences or 'runs'. These are the names for intervals \( S = \{i, i+1, \ldots; i+s-1\} \) contained in a given \( A \subseteq [n] \) such that \( S \subseteq A \) and \( i-1 \notin A, i+s \notin A \). Let \( q(A) \) be the number of runs of \( A \). Then, the number of \( a \)-blocks \( A \subseteq [n] \) with \( r \) runs \((|A| = a, \rho(A) = r)\) equals \( \binom{a-1}{r-1} \binom{n-a+1}{r} \). For the circular \( a \)-blocks with \( r \) runs, \( A \subseteq [n] \), p. 24, the number is \( \frac{n-a}{r} \binom{n-a}{r-1} \). More generally, compute the number of divisions \( A_1 + A_2 + \cdots + A_c = [n] \), where \( |A_i| = a_i \) are fixed integers \( \geq 1 \), \( i \in [c] \) and for which \( \sum_{i=1}^{c} q(A_i) = y \).

**20. Generalizations of the ballot problem** (Theorem B, p. 21.) (1) Let \( p, q, r \) be integers \( \geq 1 \), with \( q \geq rp \). Show that the number of 'minimal paths' of \( p, q \) satisfying \( y > rx \) (instead of \( y > x \) in Theorem B), equals \( \frac{q-rp}{q+p} \times \left(\frac{p+q}{q}\right) \) (for real \( > 0 \), see [Takács, 1962]). [Hint: The formula evidently holds for the points \( B(p, q) \) such that \( p = 0 \) or \( q = rp \); show next that if it holds for \( (p-1, q) \) and \( (p, q-1) \), then it holds for \( (p, q) \) as well.] (2) If in the preceding problem, the condition \( y > rx \) is replaced by \( y \geq rx \), then the number of paths becomes \( \frac{q+1-rp}{q+1} \times \left(\frac{p+q}{q}\right) \). (3) More generally, let \( P \) be the probability that a path \( \mathcal{G} \) of \( N^d \) joining \( O \) with the point \( B(p_1, p_2, \ldots, p_d) \) is such that each of its points \( M(x_1, x_2, \ldots, x_d) \) satisfies \( x_1 \leq x_2 \leq \cdots \leq x_d \) (integers \( p_i \) satisfy \( 0 \leq p_i \leq p_1 + \cdots + p_d \)). Then:

\[
P = \prod_{1 \leq i \leq d} \left(1 - \frac{P_i}{p_i + 1 - s}\right).
\]

([MacMahon, 1915], p. 133. See also [Narayana, 1959].)

21. Minimal paths with diagonal steps ([Goodman, Narayana, 1967], [Moser, Zayachkowski, 1963], [Stocks, 1967]). We generalize the concept of minimal path (p. 20) by allowing also diagonal steps. Figure 23 shows a path with 4 horizontal steps, 3 vertical steps, and 2 diagonal steps. (1) \( (q-p)/(q+p-d) \) is the probability that a minimal path with \( d \) diagonal steps joining \( O \) with \( (p, q) \) satisfies \( x < y \) (except in \( O \)). (2) The total number \( D(p, q) \) of paths (of the preceding type) going from \( O \) to \( (p, q) \) is called Delannoy number. It equals \( \sum q \binom{q+d}{d} \) or also \( \sum 2^q \binom{q}{d} \). We have \( D(p, q) = D(p, q-1) + D(p-1, q-1) + D(p-1, q) \). Hence, we get the following table of the first values of \( D(p, q) \):

<table>
<thead>
<tr>
<th>( p \times q )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>11</td>
<td>13</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>5</td>
<td>13</td>
<td>25</td>
<td>41</td>
<td>61</td>
<td>85</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>7</td>
<td>25</td>
<td>63</td>
<td>129</td>
<td>231</td>
<td>377</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>9</td>
<td>41</td>
<td>129</td>
<td>321</td>
<td>681</td>
<td>1289</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>11</td>
<td>61</td>
<td>231</td>
<td>681</td>
<td>1683</td>
<td>3653</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>13</td>
<td>85</td>
<td>377</td>
<td>1289</td>
<td>3653</td>
<td>9898</td>
</tr>
</tbody>
</table>

The GF \( \sum_{p, q \geq 0} D(p, q) x^p y^q d \) is \( (1 - x - y - xy)^{-1} \) and the diagonal series \( \sum_{n \geq 0} D(n, n) t^n \) equals \( (1 - 6t + t^2)^{-1/2} \). (3) The total number of paths joining \( O \) with \( (n, n) \), and diagonals allowed, is \( P_n(3) \), where \( P_n \) is the Legendre polynomial [141] (p. 50). (4) Let \( q_n \) be the number of paths with the property of (3) and satisfying \( x < y \) (except at the ends). Then \( (n+2) \times q_{n+1} = 3(2n+1) q_n - (n-1) q_{n-1} \). Thus show that \( q_n = 2c_n \) for \( n \geq 2 \), where \( c_n \) is the number of generalized bracketings (see p. 56).
Use Theorem B, p. 21. (3) The number of paths joining \( O \) with \((n, n)\) and such that \( x \neq y \) (except at the ends) equals \( f_n := \frac{1}{(2n-1)} \binom{2n}{n} = u_n/(2n-1) = (2/n)u_{n-1}, \ n \geq 1. \) Compute \( \sum_{n \geq 1} f_n x^n. \) (4) The number of paths starting at the origin \( O \), of length \( 2n \), and with exactly \( r \) points (different from \( O \)) on the diagonal \( x=y \) is equal to \( 2^n \binom{2n-r}{n}. \) Solve an analogous problem for the paths joining the origin \( O \) with \((p, q)\). (5) \( u_n \) and \( f_n \) are defined as in (1), (2) and (3); show that \( u_n = f_1 u_{n-1} + f_2 u_{n-2} + \ldots + f_{n-1} u_0, \ n \geq 1. \) (6) Let \( b_{n,k} \) be the number of paths of length \( 2n \) with the property that \( 2k \) segments (of the total \( 2n \)) lie above the diagonal \( x=y, \ 0 < k < n \) (in Figure 24, \( n=8, \ k=4 \)). Let the abscissa of the first passage of the diagonal (different from \( O \)) be called \( r \geq 1 \) (so, in Figure 24, \( r=3 \)). Show that:

\[
2b_{n,k} = \sum_{1 \leq r < k} f_r b_{n-r,k-r} + \sum_{1 \leq r < n-k} f_r b_{n-r,k-r}.
\]

(7) Use this to show by induction (on \( n \)) that \( b_{n,k} = u_k u_{n-k} = \binom{2k}{k} \binom{2n-2k}{n-k} \) ([Chung, Feller, 1949], [*Feller, I, 1968], p. 83). Let \( c_{n,k} \) be the number of paths of length \( 2n \) joining \( O \) with \((n, n)\) such that \( 2k \) segments lie above the diagonal. Let \( r \) be as in (6) the abscissa of first passage of the diagonal. Show that \( c_{n,k} \) does not depend on \( k \), and that it equals \( c_n = 1/(n+1). \ u_n = 1/(n+1). \ binom{2n}{n} \), a Catalan number of p. 53.

23. Multiplication table of the factorial polynomials. We consider the polynomials \((x)_m, m=0, 1, 2, \ldots, \ [4f] \) p. 6; then the product \((x)_m(x)_n \) can be expressed as a linear combination of these polynomials, and actually equals \( \sum m \binom{m}{k} n^{m-k} k! (x)_{m-k} \), where \( k \leq \min(m, n) \). [Hint: Use \( (1+t+u+ru)^x = (1+t)^x (1+u)^x \) with \([12m] \) p. 41.] Same problem for the polynomials \( (x)_{m+k} \).

24. Formal series and difference operator \( A \). (1) With the notations of \([6e] \) (p. 14) show that \( \sum_{n \geq 0} A^k(x)_n t^n/n! = e^{xt}(e^t-1)^k \) and that \( \sum_{n \geq 0} A^k(x)_n t^n/n! = e^{xt}(x-1)^k. \) (2) If \( f = \sum_{n \geq 0} f_n t^n/n! \), then, with the notations of pp. 13 and 41:

\[
\sum_{n \geq 0} (A^k f)_n t^n/n! = \sum_{k=0}^n (-1)^{n-k} \binom{k}{h} D^h f.
\]

(3) If \( f = \sum_{n \geq 0} a_n t^n \), then \( \sum_{n \geq 0} (A^k a_n) t^{n+k} = (1-t)^k f(t) \) and \( \sum_{n \geq 0} (A^n a_0) t^n = (1+t)^{-1} f((1+t)^{-1}). \)

25. Harmonic triangle and Leibniz numbers. Let us define the Leibniz numbers by

\[
\left( \begin{array}{c} n+1 \end{array} \right) (n+1)^{-1} = \binom{n+1}{k+1} (n-k)^{-1} = k! (n+1)(n-k+1) \cdots (n-k+1)\cdots (n+1)\cdots (n-k+1) \cdots (n+1) = 1/(n+1). \ u_n = 1/(n+1). \ binom{2n}{n} \), a Catalan number of p. 53.

Of course, \( L(x, k) \) could be defined for any real number.
x∈{-1, 0, 1, 2, ..., k-1} by the same manner. This "harmonic" triangle of numbers has properties very similar to those of the "arithmetic triangle" (of binomial coefficients p. 12). (1) For k≥1, \( \mathcal{L}(n, k) = \mathcal{L}(n-1, k-1) - \mathcal{L}(n-1, k-1) - \mathcal{L}(n, k-1) \). So, \( \sum_{n=1}^{\infty} \mathcal{L}(n, k) = \mathcal{L}(1, k-1) - \mathcal{L}(n, k-1) \). (2) \( \sum_{n=0}^{\infty} (-1)^{n} \mathcal{L}(n, h) = \mathcal{L}(n+1, 0) - \mathcal{L}(n+1, k+1) \). (3) \( \Delta^{k} (n^{-1}) = (-1)^{k} \mathcal{L}(n+k-1, k). \)

(4) The following GF holds:

\[
\sum_{e \in \mathcal{E}_{e}} \mathcal{L}(n, k) t^{n+1} u^{k} = -\log \left( (1 - t) (1 - u) \right) + (1 - u) (1 - t) - (1 - t)^{k} (1 - u)^{k}.
\]

So, \( \sum_{e \in \mathcal{E}_{e}} \mathcal{L}(n, k) e^{k} = \sum_{i=1}^{n} (1 + u^{1}) i^{k} / (1 - u) - \sum_{i=1}^{n} (1 + u^{n}) i^{k} / (1 - u) \) and \( \varphi_{k}^{e} = \sum_{e \in \mathcal{E}_{e}} \mathcal{L}(n, k) \mathcal{L}(n, k) t^{n-i} (1 - t)^{k-i} (1 - u)^{k-i} - \log (1 - t) \) (See Exercise 15, p. 294). (5) Let \( (n, k) \) be the "inverse" of \( \mathcal{L}(n, k) \); in other words:

\[
\varphi_{k} = \sum_{e \in \mathcal{E}_{e}} \mathcal{L}(n, k) a_{e} = \sum_{e \in \mathcal{E}_{e}} \mathcal{L}(n, k) a_{e}.
\]

Then \( \mathcal{L}(n, k) \) satisfies \( \sum_{k=0}^{\infty} (-1)^{k} \frac{1}{k!} \mathcal{L}(n, k) \mathcal{L}(n, k) = \mathcal{L}(n+1, 0) - \mathcal{L}(n+1, k+1) \). (See Exercise 16, p. 294).

26. Multisection of series. Let \( f \) be a formal series with complex coefficients, \( f = f(t) = \sum_{n=0}^{\infty} a_{n} t^{n} \) and \( \omega = \exp(2\pi i/n) \) be a \( n \)-th root of unity, \( n \) an integer > 0. Then for each integer \( u, 0 \leq u < n \):

\[
a_{n} \omega^{u} + a_{n+u} \omega^{u+u} + a_{n+2u} \omega^{u+2u} + \cdots = \frac{1}{\binom{n}{u}} \sum_{k=0}^{\infty} \omega^{-ku} f(\omega^{k} t).
\]

For example:

\[
\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots + \binom{n}{n} = \frac{1}{\binom{n}{1}} \frac{1}{\binom{n}{3}} \frac{1}{\binom{n}{5}} \cdots = 2^{n-1},
\]

\[
\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = \frac{1}{2} \left( 2^{n} + 2 \cos \left( \frac{n\pi}{3} \right) \right),
\]

\[
\binom{n}{2} + \binom{n}{4} + \binom{n}{6} + \cdots = \frac{1}{2} \left( 2^{n} + 2 \cos \left( \frac{n\pi}{2} \right) \right),
\]

\[
\binom{n}{3} + \binom{n}{5} + \binom{n}{7} + \cdots = \frac{1}{2} \left( 2^{n} + 2 \cos \left( \frac{n\pi}{3} \right) \right),
\]

and, more generally,

\[
\binom{n}{u} + \binom{n}{u+1} + \binom{n}{u+2} + \cdots = \frac{1}{\binom{n}{u}} \sum_{j=0}^{\infty} \left( 2 \cos \left( j \frac{u\pi}{n} \right) \right) \cos \left( j(n-2u) \frac{\pi}{n} \right).
\]

(See more in [Riordan, 1968], p. 131. Cf. Exercise 13, p. 76.)

27. \( p \)-bracketings. Instead of computing the products of the factors pair by pair, as on p. 52, we take now \( p \) at a time, but still adjacent. We keep \( p \) fixed > 2. Then the number \( a_{n,p} \) of these \( p \)-bracketings (\( a_{n,2} = a_{n} \) as defined on p. 52) satisfies \( a_{k(p-1)+1,p} = \binom{kp}{k-1} k \geq 1 \) and \( a_{n,p} \) is zero if \( n \) is not of the form \( k(p-1)+1 \). [Hint: \( t = y - y^{p} \), then use Lagrange formula, p. 148.]

28. A multiple sum. We sum over all systems of integers \( c_{1}, c_{2}, ..., c_{k} \geq 0 \) such that \( c_{1} + c_{2} + \cdots + c_{k} = n \), show that \( a_{n} = \sum_{c_{1} + c_{2} + \cdots + c_{k} = n} c(n)^{t}, c(0), c(1), c(2), ... = 1, -1, -1, -3, -13, -71, -461, ... \) (see Exercise 15, p. 294).

29. Hurwitz series. A formal series \( f = \sum_{n=0}^{\infty} a_{n} t^{n} \) is called a Hurwitz series if all of its coefficients are integers (\( \in \mathbb{Z} \)). When \( \mathbb{S} \) stands for the set of all such series, show the following properties: (1) \( f \in \mathbb{S} \Rightarrow Df \in \mathbb{S} \) (\( D \) and \( P \), the differentiation and primitivation operators are defined on p. 41). (2) \( f, g \in \mathbb{S} \Rightarrow f + g, f - g, fg \in \mathbb{S} \). (3) \( f, g \in \mathbb{S} \), \( g_{0} = \pm 1 \rightarrow f g^{-1} \in \mathbb{S} \). (4) \( f \in \mathbb{S} \), \( f_{0} = 0 \Rightarrow m \in \mathbb{N}, f^{m} \in \mathbb{S} \). (5) \( f, g \in \mathbb{S} \), \( g_{0} = 0 \Rightarrow f g \in \mathbb{S} \), where \( f g \) is the composition of \( g \) with \( f \) (p. 40).}

30. Hadamard product. The Hadamard product (\([\text{Hadamard}, 1893]\); see also \([\text{Benzaghou}, 1968]\)) of two formal series \( f = \sum_{n=0}^{\infty} a_{n} t^{n} \), \( g = \sum_{n=0}^{\infty} b_{n} t^{n} \) is defined by \( f \circ g := \sum_{n=0}^{\infty} a_{n} b_{n} t^{n} \) such that \( f(x, \varphi(x)) = \mathcal{S} \) is also a Hurwitz series: every \( \varphi_{e} \in \mathbb{E} \) (see [Comtet, 1968, 1974] and p. 153).
where the integration contour goes around the origin in such a way that 
\( f(z) \) is analytic on the interior, and \( g(t/z) \) is analytic on the exterior, 
\( t \) fixed and small. The symbol \( \text{CZO} \) means 'coefficient of the constant term in 
the Laurent series'. (Compare [12q], p. 42.) (3) If \( f \) and \( g \) are expansions 
of rational fractions, then \( f \circ g \) is too. Thus, for \( f := (x^2 - sx + p)^{-1}, p \neq 0, \)
we have \( f\circ g = (p + x) \left((x^2 - x + p)^{-1} (p^2 - x^2 - 2p + x^3)^{-1}\right). \) More generally, 
compute \( f \circ g \) in this case. (4) If \( f \) is rational, and \( g \) is algebraic, then \( f \circ g \) is algebraic ([Jungen, 1931], [Schützenberger, 1962]). (5) If \( f \) and \( g \) 
satisfy a differential equation with polynomial coefficients, then \( f \circ g \) does.

*31. Powers of the Fibonacci numbers. Let \( \Phi_k(t) := \sum_{n \geq 0} F_n t^n = \Phi(t)^{\circ n} \)
with the \( F_n \), p. 45 and the preceding exercise. Then:
\[
(1 - 2t - 2t^2 + t^3) \Phi_2(t) = 1 - t.
\]
[Use that \( F_n = (\phi^{n+1} - \psi^{n+1})/\sqrt{5}. \)] More generally determine explicitly 
and inductively the sequence \( \Phi_k(t). \) ([Riordan, 1962b], [Carlitz, 1962c],
[Horadam, 1965].)

32. Integers generated by \( \cosh \).
We define the Salié's integers \( S_{2n} \) by:
\[
\varphi(t) := \cosh t = \sum_{n \geq 0} S_{2n} t^{2n}.
\]
We want to show that \( S_{2n} \) is divisible by \( 2^n \). More precisely, there exist integers \( S'_{2n} \) such that
\[
S_{2n} = 2^n S'_{2n};
\]
\[
S'_{2n} = (-1)^n (\mod 4).
\]
([Carlitz, 1959, 1965c], [Gandhi, Singh, 1966]. We give the method of
[Salié, 1963].) (1) The expansion \( (\cosh t)/\cosh t = \sum S_{2n}(u) t^{2n}/(2n)! \)
defines polynomials \( S_{2n}(u) \) such that \( S_{2n} = S_{2n}(1) \), satisfying \( u^{2n} =
= \sum_k (-1)^k \binom{2n}{2k} S_{2k}(u). \) (2) Thus \( (1 + u^2)^n = \sum_{0 \leq k \leq n/2} (-1)^k \binom{n}{2k} \)
\( \times 2^k S_{2k-2k}(u). \) (3) Hence, by inversion, \( S_{2n}(u) = \sum_{k=1}^n 2^{n-k} \binom{n-k}{k} \)
\( \times (1 + u^2)^{n-k-1} \), where the \( C(n, h) \) are integers. (4) Moreover, \( C(n, 1) = 1, \)

\[
C(n, 2) = \binom{n}{2}, C(n, 3) = \binom{n}{3} \left(\binom{n-1}{2} - \frac{n}{4}\right). (5) [\#a] \text{then follows from (3),}
\]
with \( u = 1. \) (6) Hence, by (3), \( S'_{2n} = \sum_{k=1}^n 2^{n-1} C(n, h), \) so [\#b] follows.

(7) Show that \( S_{2n} = \sum_2 (2^{n-1}) \cdot E_{2n}; \) \( E_{2k} \) is an Euler number (p. 48).

33. Generating function of min. ([Carlitz, 1962a], where the GF of \( \max(n_1, n_2, \ldots, n_k) \)
also is found.) Show that:
\[
\sum_{n_1, n_2, \ldots, n_k \geq 1} \min(n_1, n_2, \ldots, n_k) t_1^{n_1} t_2^{n_2} \ldots t_k^{n_k} = \frac{t_1 t_2 \ldots t_k}{(1 - t_1)(1 - t_2) \ldots (1 - t_k)}.
\]

*34. Expansion of a rational fraction. Let \( \mathcal{R} \) be the set of rational fractions 
with complex coefficients in one indeterminate \( t \); \( f \in \mathcal{R} \) if and only if \( f = P(t)/Q(t) \)
where \( P \) and \( Q \) are polynomials, \( Q(0) \neq 0. \) Show the equivalence of the following four definitions: (1) \( \mathcal{R} \) is the set of sums \( W(z) = \sum f_{j,k} z^{j-k} / H_{j,k} \), where \( f_{j,k} \in \mathbb{C}, \) \( n_{j,k} \) integers \( \geq 1, \)
and \( E \) a finite subset of \( \mathbb{N}^2. \) (2) \( \mathcal{R} \) is the set of formal series \( \sum_{n \geq 0} a_n t^n \)
whose coefficients satisty a linear recurrence with constant coefficients 
\( c_j : \sum_{j=0}^n c_j a_{n-j} = 0, \) \( n \geq n_0. \) (3) \( \mathcal{R} \) is the set of formal series whose coefficients are of the form \( a_n = \sum_{j=1} A_j(n) \beta_j^n, n \geq n_0, \)
where the \( A_j \) are polynomials, and the \( \beta_j \neq 0. \) (4) \( \mathcal{R} \) is the set of formal series \( f = \sum_{n \geq 0} a_n t^n \)
such that for each series there exist two integers \( d \) and \( q \) for which \( H^{(d+1)}(f) = 0 \) for all integers \( j \geq 0, \) where \( H^{(d)}(f) \) are the Hankel determinants of \( f: \)
\[
\begin{vmatrix}
    a_n & a_{n+1} & \ldots & a_{n+k-1} \\
    a_{n+1} & a_{n+2} & \ldots & a_{n+k} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n+k-1} & a_{n+k} & \ldots & a_{n+2k-2}
\end{vmatrix}.
\]

35. Explicit values of the Chebyshev, Legendre and Gegenbauer polynomials.
Use \( (1 - \lambda x) \left(1 - 2x + x^2\right)^{-1} = (1 - \lambda x) \left(1 + \lambda^2\right)^{-1} (1 - 2\lambda x (1 + \lambda^2)^{-1})^{-1} \)
(p. 50) to show that $T_n(x) = \binom{n/2}{m} \sum_{m \leq n/2} (-1)^m (n-m-1)! (m!(n-2m)!)^{-1} (2x)^{n-2m}$ (compare Exercise 1, p. 155). Similarly, calculate the polynomials $U_n(x)$ and $C_n(x)$ (from which $P_n(x)$ can be obtained).

Finally, establish the following expressions with determinants of order $n$:

$$T_n(\cos \phi) = \cos n\phi = \begin{vmatrix} \cos \phi & 1 & 0 & 0 \\ 1 & 2 \cos \phi & 1 & 0 \\ 0 & 1 & 2 \cos \phi & 1 \\ 0 & 0 & 1 & 2 \cos \phi \end{vmatrix}$$

$$U_n(\cos \phi) = \frac{\sin (n + 1) \phi}{\sin \phi} = \begin{vmatrix} 2 \cos \phi & 1 & 0 & 0 \\ 1 & 2 \cos \phi & 1 & 0 \\ 0 & 1 & 2 \cos \phi & 1 \\ 0 & 0 & 1 & 2 \cos \phi \end{vmatrix}$$

36. Miscellaneous Taylor coefficients using Bernoulli numbers. Use $\tan x = (e^{2x} - 1) (e^{2x} - 1)^{-1} = 1 - 2(e^{2x} - 1)^{-1} + 4(e^{2x} - 1)^{-1}$, and [14a] (p. 48) to show that $\cot x = \sum_{m \geq 1} B_{2m} (2^{2m} - 1) x^{2m-1} / (2m)!$. From this, obtain:

$$\tan x = x - \frac{1}{3} x^3 - \frac{1}{5} x^5 - \frac{1}{7} x^7 - \cdots = x - \sum_{m \geq 1} B_{2m} (-1)^m x^{2m-1} / (2m)!$$

Thus, $\cot x = 1 / t + \sum_{k \geq 1} 2t (t^2 - n^2 \pi^2)^{-1}$

37. Using the Euler numbers. We put $\beta(s) = \sum_{n \geq 0} (-1)^n (2n+1)^{-s}$, with $s > 0$. Then, by [14c] (p. 48), and using either the Fourier expansion

$$E_{2k}(x) = 4(-1)^k (2k)! \sum_{n=0}^{\infty} \frac{\sin (2n+1) \pi x}{((2n+1) \pi)^{2k+1}}$$

or the expansion into rational functions

$$\cot t = \sum_{k \geq 1} 2t (t^2 - n^2 \pi^2)^{-1}$$

show that

$$\beta (2k + 1) = \frac{(-\pi^2)^{k+1}}{2(2k)!}.$$

Thus, $\beta(1) = \pi/4$, $\beta(3) = \pi^3/32$, $\beta(5) = 5\pi^5/1536$. 

Put now $\zeta(s) = \sum_{n \geq 1} n^{-s}$ with $s > 1$. Use either the Fourier expansion

$$B_{2k}(x) = 2(-1)^k (2k)! \sum_{n=1}^{\infty} \frac{\cos 2mnx}{(2mn)^{2k}}$$

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Put now $\zeta(s) = \sum_{n \geq 1} n^{-s}$ with $s > 1$. Use either the Fourier expansion

$$B_{2k}(x) = 2(-1)^k (2k)! \sum_{n=1}^{\infty} \frac{\cos 2mnx}{(2mn)^{2k}}$$

or the expansion in rational functions

$$\cot t = \sum_{k \geq 1} \frac{2t (t^2 - n^2 \pi^2)^{-1}}{2(2k)!}.$$
38. **Sums of powers of binomial coefficients.** For any real number \( z \), let us denote \( B(n, r) = \sum_{k=0}^{n} \binom{n}{k}^r \). Evidently, \( B(n, 0) = n+1 \), \( B(n, 1) = 2^n \), \( B(n, 2) = \left( \frac{2^n}{n} \right) \) (p. 154). (1) Prove the following recurrences: 
\[
B(n, 3) = (7n^2 - 7n + 2)B(n-1, 3) + 8(n-1)^2B(n-2, 3)
\]
and
\[
B(n, 4) = 2(2n-1)(3n^2 - 3n + 1)B(n-1, 4) + (4n-3)(4n-4)(4n-5)B(n-2, 4)
\]
([Franel, 1895]). (2) More generally, for every integer \( r \geq 0 \), we have \( B(n, r) \) (\( r \) fixed) satisfies a linear recurrence of which the coefficients are polynomials in \( n \). [Hint: (4) Exercise 30 p. 85, and [Comtet, 1964].] (3) For any real number \( b > 0 \), we have \( B(b, 2) = \sum_{k=0}^{b} \left( \frac{n}{k} \right)^2 = 2^{3b}n^{-1/2} \) (\( \beta + 1/2 \) is algebraic and \( B(n, r) \) (\( r \) fixed) satisfies a linear recurrence of which the coefficients are polynomials in \( n \). (4) Show that \( B(n, r) \) is asymptotically equal to \( n2^r - 1 \) for \( n \to \infty \). ([Comtet, 1966]).

39. **Transitive closure of a binary relation.** For two relations \( R \) and \( S \) on \( N \), the transitive product \( R \circ S \) is defined by \( x R y \Rightarrow \exists z \in N, x R z \circ z S y \). The transitive closure \( \hat{R} \) of a relation \( R \) is the 'smallest' transitive relation containing \( R \) (= the intersection of transitive relations containing \( R \)). (1) Show that \( \hat{R} = R \cup R \circ R \cup R \circ R \circ R \cup \ldots \).

40. **Forests and introductions.** We consider a graph \( G \) over \( E \) (possibly infinite), which is a forest. In other words, there exist trees \( (A_1, A_2, \ldots) \) such that \( E = A_1 + A_2 + \ldots \) and \( G = A_1 + A_2 + \ldots \).

(1) Show that \( G \) can be divided into two subsets \( V \) and \( W \), \( E = V \cup W \), such that \( V_1 \cup G \subset V \) and \( W_1 \cup G \subset W \). [Hint: Choose \( x_i \in A_i \), then divide \( A_i \) into \( V_i \cup W_i \), where \( V_i \) is the set of \( x \in A_i \) whose distance to \( x_i \) is even (p. 62); then take \( V = V_1 \cup V_2 \cup \ldots \).]

(2) In any meeting of citizens of a city \( X \), the number of necessary introductions is less than the number of people present at that meeting. Show that the population of \( X \) can be divided into two classes, such that in each of these two classes all people know each other.

41. **The pigeon-hole principle.** (1) If \( (n + 1) \) objects are distributed over \( n \) containers, then one container at least contains at least \( 2 \) objects. More generally, let \( G \) be a system of \( m \) subsets (not necessarily distinct) of \( N \), \( |N| = n = m \), such that \( \sum_{S \in G} |S| = n \). Then a sufficient condition for \( b \) points of \( N \) to be \( h \) times covered by \( G \), is \( w \geq (h-1)n + (b-1) \times (m-h+1)+1 \). (2) Let \( N \) be a set of \( n \) (\( \geq 1 \)) objects, not necessarily distinct. For one of the two following is the case: (1) \( (a+1) \) objects are identical; (II) \( (a+1) \) are distinct.

42. **Filter bases.** This is the name for a system \( G \) of \( N \), \( G \subset \mathbb{P}(N) \), such that for \( A \), \( B \in G \), there exists \( C \in G \) such that \( C \subseteq A \cap B \). The number of filter bases of \( N \), \( |N| = n \), equals \( \sum_{k=1}^{n} \binom{n}{k} 2^{k-1} \) and this is asymptotically equal to \( n2^{n-1} - 1 \) for \( n \to \infty \). ([Comtet, 1966]).

43. **Idempotents of \( \mathbb{B}(N) \) and forests of height \( \leq h \).** Let \( \mathbb{B}(N) \) be the set of maps of a finite set \( N \) into itself, \( \mathbb{B}(N) = N^N \), \( |N| = n \); \( \mathbb{B}(N) \) is also the symmetric semigroup (or monoid) of \( N \). A map \( f \in \mathbb{B}(N) \) is called idempotent if and only if for all \( x \in N \), \( f(f(x)) = f(x) \). (1) \( f \) is idempotent if and only if the restriction of \( f \) to its image \( f(N) \) is the identity. (2) The number \( z(n) \) of idempotent maps equals \( \sum_{k=1}^{n} \binom{n}{k} k^{n-k} \) ([Harris, Schenfeld, 1967], [Tainiter, 1968]).

44. **Finite geometries.** Let \( S \) be a projective space of dimension \( n \) over a finite field \( K \) (= the Galois field \( GF(q) \) of \( q = p^r \) elements, where \( p \) is a prime number. One often writes that \( S \) is a \( FG \). \( n/q \) ). \( E \) is the vector space from which \( S \) is obtained: \( \dim E = n+1 \). (1) The number of non-
zero vectors of $E$ is $q^{k+1} - 1$; use this to show that the number of points of $S$ equals $(q^{k+1} - 1)/(q-1)$. (2) The number of sets of $k + 1$ independent points (obtained from $(k+1)$ independent vectors of $E$) equals $q^{k+1} - 1 (q^{n+1} - 1) ... (q^{n-k+1} - 1) (q-1)^{k-1}$. (3) Deduce that the number of projective varieties of dimension $k$ in $S$ equals:

$$\frac{(q^{k+1} - 1) (q^n - 1) ... (q^{n-k+1} - 1))}{(q^{n+1} - 1) (q^n - 1) ... (q - 1)}.$$

(For other analogous formulas, see [*Vajda, 1967a, b*. Compare also Exercise 11, p. 118.)

**45. Bipartite trees.** Let a bipartition of a set $P$ be given, $M+N=P$ such that $m=|M|\geq 1, n=|N|\geq 1$. Show that the number of trees over $P$ such that each of $(m+n-1)$ edges of such a tree connects a point of $M$ with a point of $N$, equals $m^{n-1} n^{m-1}$. (On this subject, see [Austin, 1960], [*Berge, 1968*, p. 91, [Glicksman, 1963], [Raney, 1964], [Scoins, 1962], and especially [Knuth, 1968].)

**46. Binomial determinants.** We recall the notation $(a,b) = \binom{a+b}{a}$ (cf. p. 8). The following determinants of order $r$, taken from the table of binomial coefficients satisfy:

$$\begin{vmatrix}
\binom{n}{k} & \binom{n}{k+1} & ... & \binom{n}{k+r-1} \\
\binom{n+1}{k} & \binom{n+1}{k+1} & ... & \binom{n+1}{k+r-1} \\
\vdots & \vdots & \ddots & \vdots \\
\binom{n+r-1}{k} & \binom{n+r-1}{k+1} & ... & \binom{n+r-1}{k+r-1} \\
\end{vmatrix} = \frac{\binom{n}{k} \binom{n+1}{k} ... \binom{n+r-1}{k}}{\binom{k}{k} \binom{k+1}{k} ... \binom{k+r-1}{k}}.$$

Generalize this to determinants extracted from the table of binomial coefficients with row or column indices in arithmetic progression. (See [Zeipel, 1865] and [*Netto, 1927*, p. 256.)

**47. Equal binomial coefficients.** Determine all solutions in positive integers $u, v, x, y$ of $\binom{u}{v} = \binom{x}{y}$. Examples: $\binom{10}{3} = \binom{16}{2} = 120$, $\binom{14}{6} = \binom{15}{5} = 3003$. 
CHAPTER 11

PARTITIONS OF INTEGERS

The concept of partition of integers belongs to number theory as well as to combinatorial analysis. This theory was established at the end of the 18-th century by Euler. (A detailed account of the results up to ca. 1900 is found in [*Dickson, II, 1919], pp. 101–64.) Its importance was enhanced by [Hardy, Ramanujan, 1918] and [Rademacher, 1937a, b, 1938, 1940, 1943] giving rise to generalizations, which have not been exhausted yet. We will treat here only a few elementary (combinatorial and algebraical) aspects. For further reading we refer to [*Hardy, Wright, 1965], [*MacMahon, 1915–16], [Andrews, 1970, 1972h], [*Andrews, 1971], [Gupta, 1970], [Sylvester, 1884, 1886] (or Collected Mathematical Papers, Vol. 4, 1–83), and, for the beautiful asymptotic problems, to [*Ayoub, 1963] and [*Ostmann, 1956]. We use mostly the notations of the tables of [*Gupta, 1962], which are the most extensive ones on this matter.

2.1. Definitions of partitions of an integer n

**Definition A.** Let n be an integer \( \geq 1 \). A partition of \( n \) is a representation of \( n \) as a sum of integers \( \geq 1 \), not considering the order of terms of this sum. These terms are called summands, or parts, of the partition.

We list all partitions of the integers 1 through 5: 
- 1; 2 = 1 + 1; 3 = 2 + 1 = 1 + 1 + 1; 4 = 3 + 2 = 2 + 2 = 2 + 1 + 1; 5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1.

It is important to distinguish clearly between a partition of a set (p. 30) and a partition of an integer. But in the first case as well as in the second case, the order of the blocks and the order of the summands respectively does not play a role, and no block is empty, just like no summand equals zero.

Let \( p(n) \) be the number of partitions of \( n \), and let \( P(n, m) \) be the number of partitions of \( n \) into \( m \) summands. Thus, by the preceding list, \( p(1) = 1 \), \( p(2) = 2 \), \( p(3) = 3 \), \( p(4) = 5 \), \( p(5) = 7 \) and \( P(5, 1) = P(5, 4) = P(5, 5) = 1 \), \( P(5, 2) = P(5, 3) = 2 \). Clearly, \( p(n) = \sum_{m=1}^{n} P(n, m) \) and, since the order of the summands does not matter, we have:

**Definition B.** Each partition of \( n \) into \( m \) summands can be considered as a solution with integers \( y_i \geq 1 \), \( \text{ic } [m] \). (The summands of the partition)

\[
y_1 + y_2 + \cdots + y_m - n, \quad y_1 \geq y_2 \geq \cdots \geq y_m \geq 1.
\]

With such a partition, we can associate a minimal increasing path (in the sense of p. 30) starting from \( W(0, 1) \), with \( m \) horizontal steps and with area contained under its graph equal to \( n \). Figure 24' clarifies this idea for the partition 1 + 3 + 3 + 5 of 12. But the interpretation related to Ferrers diagram (p. 100) will turn out to be more rewarding.

**Theorem A.** Giving a partition of \( n \), in other words, giving a solution of \( [1a] \), is equivalent to giving a solution with integers \( x_i \geq 0 \) (the number of summands equal to \( i \)) of:

\[
x_1 + 2x_2 + \cdots + nx_n = n \quad \text{(also denoted by } x_1 + 2x_2 + \cdots = n).\]

If the partition has \( m \) summands, we must add to \( [1b] \) the following condition:

\[
x_1 + x_2 + \cdots + x_m = m \quad \text{(also denoted by } x_1 + x_2 + \cdots = n).\]

\[\text{Evident.}\]

If \((x_1, x_2, \ldots)\) are the nonzero \( x_i \) in \( [1b] \), we call the
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The corresponding partition "the partition with specification \(i_1^x_1, i_2^x_2, \ldots\)", omitting the exponents \(x_i\) which equal 1. Written in this way, the partitions of 5 become 5, 12, 13, 112, 22, 23, 31, 32, 33, 212, 213.

We write \(p(n, m)\) for the number of partitions of \(n\) with at most \(m\) summands, or also 'distribution function' of the number of partitions of \(n\) with respect to the number of summands, \(p(n, m) = \sum_{k=1}^n P(n, k)\), \(P(n, m) = p(n, m) - p(n, m-1)\). (The analogy with a stochastic distribution function will be noted.)

**Theorem B.** If \(m > n \geq 1\), then \(p(n, m) = p(n)\), and for \(n \geq m \geq 2\):

\[ p(n, m) = p(n, m-1) + p(n-m, m); \quad p(n, 1) = 1, \quad p(0, m) = 1. \]

\(p(n, m)\) is the number of solutions of \([1b]\) that satisfy \(x_1 + x_2 + \cdots \leq m\) also. So we divide the set of solutions into two parts: first the solutions of \([1b]\) that also satisfy \(x_1 + x_2 + \cdots \leq m - 1\); there are \(p(n, m-1)\) of these; then the solutions of \([1b]\) which also satisfy \(x_1 + x_2 + \cdots = m\); these are just the solutions of \(x_2 + x_3 + \cdots = n - m\) and \(x_2 + x_3 + \cdots \leq m\) (since \(x_1 \geq 0\)); hence there are \(p(n-m, m)\) of these.

The following table shows the first values of \(p(n, m)\) (boldface printed: \(p(n)\)). (See also [*Gupta, 1962], \(n \leq 400, m \leq 50\). For a table of \(p(n, m)\) and \(p(n)\) see p. 307.)

| \(m\) \(n\) \(0\) \(1\) \(2\) \(3\) \(4\) \(5\) \(6\) \(7\) \(8\) \(9\) |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   |
| 2   | 1   | 1   | 2   | 1   | 2   | 1   | 2   | 1   | 2   |
| 3   | 1   | 2   | 3   | 3   | 3   | 2   | 3   | 2   | 3   |
| 4   | 1   | 2   | 3   | 5   | 4   | 5   | 4   | 5   | 4   |
| 5   | 1   | 2   | 3   | 5   | 7   | 7   | 6   | 7   | 6   |
| 6   | 1   | 2   | 3   | 5   | 7   | 10  | 9   | 8   | 7   |
| 7   | 1   | 2   | 3   | 5   | 7   | 10  | 13  | 12  | 11  |
| 8   | 1   | 2   | 3   | 5   | 7   | 11  | 15  | 14  | 13  |
| 9   | 1   | 2   | 3   | 5   | 7   | 11  | 15  | 15  | 14  |

2.2. Generating functions of \(p(n)\) and \(P(n, m)\)

**Theorem A.** The generating function of the number \(p(n)\) of partitions of \(n\) equals:

\[ \Phi(t) := 1 + \sum_{n \geq 0} p(n) t^n = \prod_{i=1}^\infty \frac{1}{(1 - t^i)^{-1}} = \frac{1}{(1 - t)(1 - t^2)(1 - t^3) \ldots} \]

\(\Phi(t)\) is indeed the coefficient of \(t^n\) in \([2b]\) is just the number of solutions of \([1b]\) p. 95, hence \(p(n)\).

One could prove that \(\Phi(t)\), written in the form \([2a]\) as a series or as an infinite product, is convergent for \(|t| < 1\).

The family of formal series \(u_1 \cdot (1 - t)^{-1} = \sum_{n \geq 1} u_1^n t^n = (1 + t + t^2 + \cdots) = \prod_{i=1}^\infty (1 - u_1 t^i) = \prod_{i \geq 1} \frac{1}{1 - u_1 t^i}\).

By this product and this proves that the coefficient of \(t^n\) in \([2b]\) is just the number of solutions of \([1b]\) p. 95, hence \(p(n)\).

For given integer \(n\), the actual computation of \(p(n)\) by \([2a]\) is evidently performed by just considering the finite product \(\prod_{i=1}^{\infty} (1 - t^i)^{-1}\).

**Theorem B.** The generating function of the number \(P(n, m)\) of the partitions of \(n\) into \(m\) summands equals:

\[ \Phi(t, u) := 1 + \sum_{1 \leq n \leq m} P(n, m) t^u = \prod_{i=1}^\infty \frac{1}{(1 - ut^i)^{-1}} = \frac{1}{(1 - ut)(1 - ut^2)(1 - ut^3) \ldots} \]

As in the preceding proof, we have:

\[ \prod_{i \geq 1} (1 - ut^i)^{-1} = \prod_{i \geq 1} \left( \sum_{x_i \geq 0} u^{x_i} t^{x_i} \right) = \sum_{x_1, x_2, \ldots \geq 0} \left( t^{x_1 + 2x_2 + \cdots} \right) \]

Hence indeed the coefficient of \(t^n u^m\) in \([2d]\) equals the number of solutions of \([1b, c]\) (p. 95).
2.3. Conditional partitions

More generally, let \( p(n \mid \mathcal{P}_1, \mathcal{P}_2) \) be the number of partitions of \( n \) such that the number of summands has the property \( \mathcal{P}_1 \), and the value of each summand has the property \( \mathcal{P}_2 \); we indicate by a star * the absence of a condition (notations from [*Ayoub, 1963*], p. 193). Thus, \( p(n, m) = p(n \mid m, *) \), \( P(n, m) = p(n \mid m, *) \). We also denote the number of partitions of \( n \), that satisfy \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) in the sense above, and whose summands are all unequal, by \( q(n \mid \mathcal{P}_1, \mathcal{P}_2) \). Thus, \( q(n \mid *, < r) \) is the number of partitions of \( n \) into unequal summands, that are all \( \leq r \).

**Theorem A.** Let:

\[
\Xi(t, u) := \sum_{n,m \geq 0} p(n \mid m, \mathcal{P}) r^m u^n;
\]

then:

\[
\begin{align*}
\sum_{n \geq 0} p(n \mid *, \mathcal{P}) r^n &= \Xi(t, 1) \\
\sum_{n \geq 0} p(n \mid even, \mathcal{P}) r^n &= \frac{1}{2} [\Xi(t, 1) + \Xi(t, -1)] \\
\sum_{n \geq 0} p(n \mid odd, \mathcal{P}) r^n &= \frac{1}{2} [\Xi(t, 1) - \Xi(t, -1)] \\
\sum_{n,m \geq 0} p(n \mid \leq m, \mathcal{P}) r^m u^n &= (1-u)^{-1} \Xi(t, u).
\end{align*}
\]

Analogous inequalities hold when everywhere in [3a, b, c, d, e] if \( p \) is replaced by \( q \).

**[3b]** follows from \( p(n \mid *, \mathcal{P}) = \sum_{m \geq 0} p(n \mid m, \mathcal{P}) \); **[3c]** from \( p(n \mid even, \mathcal{P}) = \sum_{m \geq 0} p(n \mid 2m, \mathcal{P}) \); **[3d]** from \( p(n \mid odd, \mathcal{P}) = \sum_{m \geq 0} p(n \mid 2m+1, \mathcal{P}) \); **[3e]** from \( p(n \mid \leq m, \mathcal{P}) = \sum_{n=0}^m p(n \mid i, \mathcal{P}) \).

**Theorem B.** Let \( \mathcal{A} \) be an infinite matrix consisting of 0 and 1, \( \mathcal{A} = [a_{i,j}], i \geq 1, j \geq 0, a_{i,j} = 0 \) or 1. Denoting by \( p(n \mid m, \mathcal{A}) \) the number of partitions of \( n \) into \( m \) summands such that the number of summands equal to \( i \), equals one of the integers \( j \geq 0 \) for which \( a_{i,j} = 1 \). Then we have:

\[
\sum_{n,m \geq 0} p(n \mid m, \mathcal{A}) r^m u^n = \prod_{i \geq 1} \left( \sum_{x \geq 0} a_{i,x} u^x t^x \right).
\]

where the (bound) variable \( x \) takes only integer values.

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- The number of partitions of the indicated kind is equal to the number of solutions with integer \( x_i \geq 0, i-1, 2, \ldots \), of:

\[
\begin{align*}
[3g] & \quad x_1 + 2x_2 + \cdots = n, \quad \ x_1 + x_2 + \cdots = m, \\
& \quad x_i \in \{ j \mid j \geq 0, a_{i,j} = \} \quad (\iff a_{i,m} = 1).
\end{align*}
\]

Now, the right-hand member of [3f] can be written:

\[
\prod_{i \geq 1} \left( \sum_{x \geq 0} a_{i,x} u^x t^x \right) = \sum_{x_1, x_2, x_3, \ldots} a_{1,x_1} a_{2,x_2} \cdots u^{x_1+x_2+\cdots} t^{x_1+2x_2+\cdots},
\]

which proves that the coefficient of \( u^m t^n \) is just equal to the number of solutions of [3g].

For example, if \( a_{i,0} = a_{i,1} = 1 \) and \( a_{i,j} = 0 \) if \( j > 1 \), then we have

\[
[3h] \quad p(n \mid m, \mathcal{A}) = Q(n, m) = \text{the number of partitions of } n \text{ into } m \text{ unequal summands}; \quad \Xi(t, 1) = 1 + \sum_{n,m \geq 1} p(n \mid m, \mathcal{A}) r^m u^n = \prod_{i \geq 1} (1 + u^i).
\]

Similarly, with \( q(n) = \text{the number of partitions of } n \text{ into unequal summands} = \sum_{n,m \geq 1} Q(n, m) \), we obtain with [3b, h]:

\[
[3i] \quad q(n) = 1 + \sum_{n \geq 1} q(n) (r) = \prod_{i \geq 1} (1 + i^i).
\]

**Here are a few values of \( q(n) \):**

\[
\begin{array}{cccccccccccccccccccc}
 q(n) & 1 & 1 & 2 & 2 & 3 & 4 & 5 & 6 & 8 & 10 & 12 & 15 & 18 & 22 & 27 & 32 & 38 & 46 & 54 & 64 & 76 & 89 \\
\end{array}
\]

With the same method, or otherwise, we get also:

\[
[3j] \quad 1 + \sum_{n,m \geq 1} p(n \mid m, \leq l) r^m u^n = \prod_{i \leq l} (1 - u^i)^{-1}
\]

\[
[3k] \quad 1 + \sum_{n,m \geq 1} q(n \mid m, \leq l) r^m u^n = \prod_{i \leq l} (1 + u^i).
\]

### 2.4. Ferrers diagrams

A convenient and instructive representation of a partition of \( n \) into summands \( y_i \), such that \( [1a] \) p. 95 consists of a figure having \( m \) horizontal rows
of points (the lines), the bottom one having \( y_1 \) points, the next to bottom one having \( y_2 \) points, etc., in such a way that the initial points of every line are all on one vertical line; hence the number of points on every vertical line or column decreases going from left to right. Such a figure, the **Ferrers diagram (or relation)**, clearly determines a unique partition of \( n \). For example, Figure 25 shows the diagram of the partition \( 6 + 5 + 5 + 2 + 2 + 1 \) of 21. If one considers the **columns** from left to right, the number of points in these will constitute another partition of \( n \), with summands \( z_1, z_2, \ldots \), which is called the **conjugate** partition of the partition with summands \( y_1, y_2, \ldots \). In the case of the figure shown, the conjugate partition is \( 6 + 5 + 3 + 3 + 3 + 1 \). Certain properties of a partition \( y_1 + y_2 + \cdots \) have an immediate translation into terms of the conjugate partition. Thus we have:

**THEOREM A.** The number of partitions of \( n \) into at most (or exactly) \( m \) summands is equal to the number of partitions of \( n \) into summands that are all \( \leq m \) (or whose maximum is \( m \)), in other words, the number of partitions of \( n \pm m \) whose maximum summand equals \( m \).

**THEOREM B.** The number of partitions of \( n \) into unequal odd summands equals the number of ‘self-conjugate’ partitions of \( n \) (that is, whose diagram is symmetric with respect to the line \( x = y \)).

Theorem A is evident. For Theorem B, we associate with every partition \( (2z_1 - 1) + (2z_2 - 1) + \cdots = n \), where \( z_1 > z_2 > \cdots \), the partition whose diagram is obtained by ‘folding’ the rows of the original diagram in the middle, so they form the sides of isosceles straight-angled triangles, and fitting them then one by one, beginning with the largest, into each other. For instance, Figure 26 corresponds to \( 11 + 7 + 3 + 1 = 6 + 5 + 4 + 4 + 2 + 1 \).
result of MacMahon states ([MacMahon, II, 1916], p. 171):
\[
[4c] \quad \sum_{n \geq 0} p_3(n) t^n = \prod_{i \geq 1} (1 - (1 - i)^{-i}),
\]
but the proof is very difficult ([Chauky, 1931, 1932]). No other simple GF for \(d \geq 4\) is known. ([Atkin, Bratley, Macdonald, Mackay, 1967].
See also [Gordon, Houten, 1968], [Stanley, 1972], [Stanley, 1971a, b], [Wright, 1965a].)

2.5. Special identities; 'formal' and 'combinatorial' proofs

First we prove two typical identities, which may serve as sample of many others.

THEOREM A. The formal series introduced in [2a, c] (p. 97) also satisfy:

\[
[5a] \quad \Phi(t) = 1 + \sum_{n \geq 0} p(n) t^n = \prod_{i \geq 1} (1 - (1 - i)^{-1})
\]

\[
= 1 + \sum_{m \geq 1} \frac{1}{(1 - t)(1 - t^2) \ldots (1 - t^m)}
\]

\[
[5b] \quad \Phi(t, u) = 1 + \sum_{1 \leq \alpha \leq n} P(n, \alpha) u^\alpha = \prod_{i \geq 1} (1 - u^i)^{-1}
\]

In the literature, often \(t = q\) and \(u = x\) (in honour of the elliptic functions); hence the name of 'q-identity', often given to this kind of identity. (See also Exercise 11, p. 118).

Formal proof (also called 'algebraic' proof). We expand \(\Phi(t, u)\) in a formal series in \(u\):

\[
[5c] \quad \Phi(t, u) = \sum_{m \geq 0} C_m u^m, \quad C_m = C_m(t).
\]

The evident functional relation \(\Phi(t, tu) = (1 - tu)\Phi(t, u)\), which is satisfied by \(\Phi(t, u) = \prod_{i \geq 0} (1 - ut^i)^{-1}\), gives, when [5c] is substituted into it:
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\[ \sum_{n \geq 0} C_n t^n u^m = (1 - tu) \sum_{n \geq 0} C_n u^n. \]

If we compare the coefficients of \( u^n \) of both members of \([5d]\), we get
\[ t^n C_n = C_{n-1} - t C_{n-2}; \]


\[ C_n = \frac{1}{1-t} C_{n-1} - \frac{t^2}{(1-t) (1-t^{-1})} C_{n-2} - \ldots \]

\[ = \frac{1-t^n}{(1-t) (1-t^{-1}) \ldots (1-t^2) (1-t)}, \]

which, by \([5c]\), proves \([5b]\). By putting \( u = 1 \) we get \([5a]\).

**Combinatorial proof.** As an example we prove \([5a]\). By \([3j]\) (p. 99), the coefficient of \( t^n \) in \( \left( (1-t) (1-t^2) \ldots (1-t^r) \right)^{-1} \) equals \( p(k \mid *, \leq l) \), which is the number of partitions of \( k \) into summands smaller or equal to \( l \), here denoted by \( s(k, l) \). Hence, for proving \([5a]\), we just have to verify that the coefficients of \( t^n \) on both sides are equal; this means that we must prove that:

\[ p(n) = s(n-1, 1) + s(n-2, 2) + \ldots. \]

By Theorem A (p. 98) \( s(k, l) \) equals the number \( r(k+l, l) \) of partitions of \( k+l \) whose largest summand equals \( l \). So \([5f]\) is equivalent to \( p(n) = r(n, 1) + r(n, 2) + \ldots \) and this last equality follows from the division of the set of partitions of \( n \) according to the value of the largest summand. \( \blacksquare \)

**Theorem B.** (Sometimes called 'pentagonal theorem' of Euler). We have the following identity \([5g]\) between formal series and the recurrence relation \([5h]\) between the \( p(n) \):

\[ \prod_{i \geq 1} (1-t^i) = \sum_{k \geq 2} (-1)^{i-1} t^{i(3k+1)/2} = 1 + \sum_{k \geq 1} (-1)^{k-1} \left( \frac{k(3k-1)}{2} + \frac{t^{3k+1}}{2} \right) \]

\[ p(n) = p(n-1) + p(n-2) - p(n-5) - \ldots = \sum_{k \geq 1} (-1)^{k-1} \left( p\left( n - \frac{k(3k-1)}{2} \right) + p\left( n - \frac{k(3k+1)}{2} \right) \right). \]

**Formal proof.** Use the Jacobi identity, which is Theorem D below, and Exercise 14 (1) (p. 119).

**Combinatorial proof.** By using \([3h]\) for \((***)\), and the notations of Theorem C (p. 101), for \((***)\), we get:

\[ \prod_{i \geq 1} (1-t^i)^{(n)} \Psi(t, -1)^{(n)} = 1 + \sum_{n \geq 1} \{ p(n) \} t^n, \]

and thus \([5g (**)\) follows from \([4a]\). For \([5h]\), substitute \([5g]\) into \([5i]\) (which is equivalent to \([2a]\), p. 97):

\[ \prod_{i \geq 1} (1-t^i) \Psi(t, -1)^{(n)} = 1 + \sum_{n \geq 1} \{ p(n) \} t^n = 1, \]

and by observing that the coefficient of \( t^n (n \geq 1) \) of the left-hand member equals 0, we obtain the result. \( \blacksquare \)

**Theorem C.** The number of partitions of \( n \) into \( m \) unequal summands equals the number of partitions of \( n - \frac{(m+1)(m+2)}{2} \) into at most \( m \) summands (that is, into summands which are all \( \leq m \), by Theorem A, p. 100):

\[ Q(n, m) = p\left( n - \frac{(m+1)(m+2)}{2} \right) \]

\[ = p\left( n - \frac{(m+1)}{2} \mid *, \leq m \right). \]

**Formal proof.** This is carried out by a method analogous to the method used in the formal demonstration of Theorem A (p. 103), but this time the functional relation \( \Psi(t, u) = (1 + tu) \Psi(t, tu) \) is used. We get:

\[ \prod_{i \geq 1} (1-t^i)^{(n)} \Psi(t, u)^{(m)} = 1 + \sum_{n \geq 1} Q(n, m) t^n u^m = \prod_{i \geq 1} (1 + tu)^{(m+1)} = 1 + \sum_{n \geq 1} \frac{t^m}{(1-t) (1-t^2) \ldots (1-t^n)}. \]

Hence, \( Q(n, m) \) equals the coefficient of \( t^n (m+1) \) in \( \left( (1-t) (1-t^2) \ldots (1-t^m) \right)^{-1} \), which is \( p\left( n - \frac{(m+1)}{2} \mid *, \leq m \right) \), because of \([3j]\) p. 99, hence equal to \( p\left( n - \frac{(m+1)}{2} \right) \), by Theorem A (p. 100).
Combinatorial proof. The number of solutions of

\[ n = y_1 + y_2 + \cdots + y_m = n, \quad y_1 > y_2 > \cdots > y_m \geq 1 \]

is evidently equal to \( Q(n, m) \). We put

\[ z_1 := y_1 - y_2 - 1, \ldots, z_{m-1} := y_{m-1} - y_m - 1, \quad z_m := y_m - 1. \]

Hence \( y_1 = 1 + z_1, \quad y_{m-1} = 1 + z_1 + z_2 + \cdots + z_{m-1}, \quad y_m = 1 + z_1 + z_2 + \cdots + z_m \). Then equation [5l] is equivalent to:

\[ \sum_{i \in [m]} z_i \geq 0, \quad i \in [m]. \]

Now, the number of solutions of [5m] is clearly equal to the number of partitions of \( n - \binom{m+1}{2} \) into summands not exceeding \( n \), in other words, \( p(n - \binom{m+1}{2}, n) \), by Theorem A (p. 100).

**Theorem D. (Jacobi identity):**

\[ \prod_{i > 0} \left( 1 + t^{2i+1} \right) (1 + t^{2i+1}u) \left( 1 + t^{2i+1}u^{-1} \right) = \sum_{n \in \mathbb{Z}} t^n u^n. \]

Both sides of [5n] have a generalized formal series in \( u \), with positive and negative exponents: the theory of such series is easily developed, as on p. 43. We give here the 'formal' proof of [Andrews, 1965]. A beautiful 'combinatorial' proof is found in [Wright, 1965b]. See also [*Hermite, Oeuvres, Vol. II, pp. 155-56, and [Stolarsky, 1969].]

We replace \( tu \) by \( u \) in [5k], and \( tu \) by \( -u \) in [5b]. Then we get:

\[ \prod_{i > 0} \left( 1 + tu \right) = \sum_{n > 0} \frac{t^{n-1}}{(1-t)(1-t^2)\cdots(1-t^n)} \]

\[ \prod_{i > 0} \left( 1 + tu \right)^{-1} = \sum_{n > 0} \frac{(-1)^n}{(1-t)^{(1-t^2)\cdots(1-t^n)}}. \]

It follows (justifications at the end) that:

\[ \prod_{i > 0} \left( 1 + t^{2i+1}u \right) = \frac{1}{\prod_{n > 1} (1 - t^{2n-1})(1 - t^{2n-4})} \]
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\[ 1 + \sum_{n \geq 1} \frac{t^{p(n+1)}}{(1-t)(1-t^2) \cdots (1-t^n)} = 1 + \sum_{n \geq 1} \frac{1}{(1-t^{2n-2})(1-t^{5n-3})}. \]

\[ \text{(See also Exercises 9, 10, 11 and 12, p. 117.)} \]

2.6. Partitions with forbidden summands; denumerants

Now we consider partitions of \( n \) whose summands are taken (repetitions allowed) from a sequence of integers \( (a_1, a_2, \ldots) \), \( 1 \leq a_1 < a_2 < \cdots \).

As in Theorem A (p. 95), giving such a partition is equivalent to giving a solution of

\[ a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots = n, \quad x_i \text{ integer } \geq 0. \]

In other words, the matrix \( \mathbb{A} = [a_{i,j}] \) (p. 98) is such that \( a_{i,j} = 1 \) for \( i \in (a) \), for all \( j \geq 0 \), and \( a_{i,j} = 0 \) otherwise, except that \( a_{0,0} = 1 \). From Theorem B (p. 98) (or by direct computation) it follows immediately that:

**Theorem A.** The generating function of the number \( D(n; (a)) = \sum D(n; a_1, a_2, \ldots) \) of solutions of \( [6a] \), called the denumerant of \( n \) with respect to the sequence \((a)\), equals:

\[ D_{\{a\}}(t) := 1 + \sum_{n \geq 1} D(n; (a)) t^n = \prod_{i \geq 1} (1 - t^{a_i})^{-1}. \]

For \((a) = \{1, 2, 3, \ldots\}\), we find back \([2a]\) of p. 97.

For example, in the money changing problem, one has as many coins of 5, 10, 20 and 50 centimes as one needs. In how many ways can one make with these a given amount of, say, 15 francs? (1 franc = 100 centimes). This is equivalent to finding the number of integer solutions of \( 5x_1 + 10x_2 + 20x_3 + 50x_4 = 500 \), or equivalently, of \( x_1 + 2x_2 + 4x_3 + 10x_4 = 100 \). The solution is hence \( D(100; 1, 2, 4, 10) \), which is 2691 (see p. 113).

Another example: it is immediately clear, by \([5b]\) p. 103 and \([6b]\) that

\[ D(n; 1, 2, 3, \ldots k) = P(n + k, k) = Q(n + k (k + 1)/2, k). \]

We investigate the case of a finite sequence \((a) := (a_1, a_2, \ldots, a_k)\),

\[ 1 \leq a_1 < a_2 < \cdots < a_k \quad (\Rightarrow a_1 = 0, \text{if } k > l). \]

The GF \([6b]\) is then a rational fraction:

\[ \text{[6b']} D_{\{a\}}(t) = 1 + \sum_{n \geq 1} D(n; (a)) t^n = \prod_{i=1}^k (1 - t^{a_i})^{-1}. \]

A first method for computing the denumerant \( D(n; (a)) \) is provided by a decomposition of the rational fraction \([6b']\) into partial fractions.

For instance:

\[ D_{\{1, 2\}}(t) = \frac{1}{(1-t)(1-t^2)} = \frac{1}{4} \left( \frac{1}{1+t} + \frac{1}{1-t} + \frac{2}{1-t^2} \right) = \frac{1}{4} \left( \sum_{n \geq 0} (-t)^n + \sum_{n \geq 0} t^n + 2 \sum_{n \geq 0} (n+1) t^n \right), \]

which gives as coefficient of \( t^n \):

\[ \text{[6c]} D(n; 1, 2) = \frac{1}{4} \{2n + 3 + (-1)^n\}. \]

Similarly, for \( D_{\{1, 2, 3\}}(t) = \{(1-t)(1-t^2)(1-t^3)\}^{-1} \) we have two decompositions. (The first one, called the first type, is a decomposition into ordinary partial fractions; the second one is called the second type or Herschelian type. See [Herschel, 1818].)

\[ \text{[6d]} D_{\{1, 2, 3\}} = \frac{1}{6(1-t)^3} + \frac{1}{4(1-t)^2} + \frac{1}{17} \frac{1}{72(1-t)} + \frac{1}{8(1+t)} + \frac{1}{9(1+t+t^2)} + \frac{2 + t}{72(1-t) + 8(1+t) + 9(1+t+t^2)} = \]

\[ = \frac{1}{6(1-t)^3} + \frac{1}{4(1-t)^2} + \frac{1}{4(1-t^2)} + \frac{1}{3(1-t^3)}. \]

We denote the periodic sequence with period \( T \) (integer \( \geq 1 \)), that is equal to \( d_i \) for \( n \equiv i \pmod{T} \), \( i = 0, 1, \ldots, T-1 \), by: \( (d_0, d_1, \ldots, d_{T-1}) \) \( \text{cr} T \), \( \text{cr} \) for *circulator, this notation is from Herschel*). If, moreover, for each divisor \( S \) of \( T \), \( 1 \leq S \leq T \), we have \( d_n + d_{n+S} + d_{n+2S} + \cdots + d_{n+rS} = 0 \) for all \( R = 0, 1, 2, \ldots, S-1 \), then we rather denote the above sequence by \( (d_0, d_1, \ldots, d_{T-1}) \) \( \text{pcr} T \), \( \text{pcr} \) stands for *prime circulator, the notation is due to Cayley*. The expansion of \([6d]\) into a
power series gives then the following two forms for \( D(n; 1, 2, 3) \):

\[
\begin{align*}
[6c] \quad & \frac{n^2}{12} + \frac{n}{2} + \frac{47}{72} + \frac{1}{8} (1, -1) \text{ per } 2_n + \frac{1}{9} (2, -1, -1) \text{ per } 3_n \\
[6e'] \quad & \frac{1}{12} (n + 1) (n + 5) + \frac{1}{4} (1, 0) \text{ per } 2_n + \frac{1}{3} (1, 0, 0) \text{ per } 3_n.
\end{align*}
\]

For each \( x (\in \mathbb{R}) \) such that \((x - \frac{1}{2})\) is not integer, we put:

\[ [6f] \quad \|x\| := \text{the integer closest to } x. \]

By \[6c\] we find:

\[ [6g] \quad D(n; 1, 2) = \left\lfloor \frac{2n + 3}{4} \right\rfloor. \]

A similar formula using \[\|...\|\] for \( D(n; 1, 2, 3) \) can be found as it follows. We transform \[6e'\] by grouping first the two \( cr \)'s, then replacing \((n + 1) (n + 5)\) by \((n + 3)^2 - 4\):

\[ D(n; 1, 2, 3) = \frac{1}{12} \{(n + 1) (n + 5) + (7, 0, 3, 4, 3, 0) \text{ cr } 6_n\} = \frac{1}{12} \{(n + 3)^2 + (3, -4, -1, 0, -1, -4) \text{ cr } 6_n\}. \]

Now, \( \varphi(n) := \frac{1}{12} (3, -4, -1, 0, -1, -4) \text{ cr } 6_n \), a sequence of period 6, satisfies \( |\varphi(n)| \leq \frac{1}{2} \). Hence,

\[ [6g'] \quad D(n; 1, 2, 3) = \|\frac{1}{12}(n + 3)^2\|. \]

This way of writing by means of \[\|...\|\] is not unique. In the same way one will find for \( D(n; 1, 2, 3) \) slightly more complicated formulas \[\|\frac{1}{12}(n^2 + 6n + 7)\|, \|\frac{1}{12}(n + 2) (n + 4)\|\] and \[\|\frac{1}{12}(n^2 + 6n + 10)\|. \]

The first values of \( D(n) = D(n; 1, 2, 3) \) are:

\[
\begin{array}{cccccccccccc}
D(n) & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 10 & 12 & 14 & 16 & 19 & 21 & 24 & 27 & 30 & 33 & 37 & 40 & 44 \\
\end{array}
\]

The following is a second method for computing the denominator.

**Theorem B.** ([Bell, 1943]). Let \( A \) be the least common multiple of the integers \((a_1, a_2, \ldots, a_k), 1 \leq a_1 < a_2 < \cdots < a_k\). For every integer \( B \) such that \( 0 \leq B \leq A - 1 \), and every integer \( m \geq 0 \), we have:

\[ [6b] \quad D(Am + B; a_1, a_2, \ldots, a_k) = D(Am + B; (a)) = D(Am + B) = \delta(m) = c_0 + c_1 m + \cdots + c_k - 1 m^{k-1}, \]

where the \( c_i, i + 1 \in [k] \), are constants independent of \( m \), and where the denominator \( \delta(m) \) is as defined as in Theorem A (p. 108).

Let \( a \) be the complex number such that \( a^A = 1 \), \( \arg a = 2\pi / A \); then we put, with \( D(n) := D(n; a_1, \ldots, a_k) \):

\[ [6i] \quad A := a_j^k, \quad j \in [k]; \quad P(t) := \prod_{1 \leq j \leq k} (1 - t^{a_j}). \]

The roots of \( P(t) = 0 \) are hence of the form \( x^w a_j \), where \( m_j + 1 \in [a_j] \) and \( j \in [k] \). Let \( \varepsilon_0 = (1), \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r \) be the \( (r + 1) \) different values of these roots:

\[ [6j] \quad \nu(t) = \left(\frac{1 - t}{\varepsilon_0}\right)^{l} \left(\frac{1 - t}{\varepsilon_1}\right)^{l_1} \cdots \left(\frac{1 - t}{\varepsilon_r}\right)^{l_r}.
\]

Identifying in \[6k\] the coefficients of \( t^n \), calculated by using the expansion of \((1 - T)^{-N}\) in the right-hand member (see [12e'], p. 37), we obtain:

\[ [6l] \quad D(n) = \sum_{0 \leq v \leq \nu} C_{v, r} \left(\frac{\nu}{r}\right)^{v} e^{-n}. \]

If we put \( n = Am + B \) in \[6i\], we get by using \( e^1 = 1 \):

\[ [6m] \quad \delta(m) = \sum_{1 \leq e \leq k} P_e(m) \left(\sum_{0 \leq v \leq \nu} C_{v, r} e^{-v}ight), \]

where the polynomial \( P_e(m) = \nu \cdot (m + 1) \cdot (1/(v - 1)) \) is the product of \((v - 1)\) factors of the first degree in \( m \) (because \( n = Am + B \)), and hence of degree \((v - 1) (\leq k - 1)\); \[6k\] follows. ■
The polynomial $\delta(m)$, of degree $(k-1)$ in $m$, is known by [6h], when the values $\delta(m_j)$ are known in $k$ different points $m_j$, $i\in[k]$. For this, we can use either the determinant $[6n]$, of order $(k+1)$, which eliminates the constants $c_i$, $(j+1)\in[k]$, from $[6h]$: 

$$
\begin{vmatrix}
\delta(m) & 1 & m & m^2 & \cdots & m^{k-1} \\
\delta(m_1) & 1 & m_1 & m_1^2 & \cdots & m_1^{k-1} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\delta(m_k) & 1 & m_k & m_k^2 & \cdots & m_k^{k-1}
\end{vmatrix} = 0,
$$

or the Lagrange interpolation formula:

$$
\delta(m) = \sum_{i=1}^{k} \delta(m_i) \mu_i,
$$

where

$$
\mu_i = \frac{(m-m_1) \cdots (m-m_{i-1})(m-m_{i+1}) \cdots (m-m_k)}{(m_i-m_1) \cdots (m_i-m_{i-1})(m_i-m_{i+1}) \cdots (m_i-m_k)}.
$$

Particularly, for $m=i$, $i\in[k]$, $[6o]$ becomes:

$$
\delta(m) = \binom{m-1}{k} \sum_{1 \leq i \leq k} (-1)^{k-1} \binom{k}{i} \frac{i}{m-i} \delta(i).
$$

For example, to calculate $D(n; 1, 2, 4):=D(n)$ by means of $[6p]$, one may use the first values of $D(n)$ (computed from $D(n; 1, 2, [6c])$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D(n)$</td>
<td>4</td>
<td>6</td>
<td>6</td>
<td>9</td>
<td>9</td>
<td>9</td>
<td>12</td>
<td>12</td>
<td>16</td>
<td>16</td>
<td>20</td>
<td>20</td>
</tr>
</tbody>
</table>

This gives $D(4m) = D(4m+1) = (m+1)^2$, $D(4m+2) = D(4m+3) = (m+1)(m+2)$. It is then verified that:

$$
D(n; 1, 2, 4) = \|[(n+2)(n+5) + (-1)^n n]/16\|.
$$

We now show, by two examples, an efficient practical use of Theorem B, without decomposition of rational fractions into partial fractions, which works particularly well if the LCM $A$ of $(a_1, a_2, a_3, \ldots)$ is not large. We abbreviate $x_s = (1-t^s)^{-1}$ and we use a point (for saving place) to denote the center of symmetry of any reciprocal polynomial. (Examples: $1+t+t^2+\cdots = 1+t+t^2+t^3$, $1+2t+\cdots = 1+2t+2t^2+t^3$).

(1) We return to $D(n; 1, 2, 3)$, [6d] (p. 109). We have

$$
D_1, 2, 3 = x_1 x_2 x_3 = (1+t) x_1^2 x_3 = (1+t) (1+t^2 + t^4)^2 (1+t^2) x_3^2 = (1+t+2t^2+3t^3+4t^4+5t^5+4t^6+5t^7+\cdots) \sum \binom{m+2}{2} t^m.
$$

Hence, identifying the coefficients in the first and last member, we get:

$$
D(6m+B; 1, 2, 3) = a\binom{m+2}{2} + b\binom{m+1}{2} + c\binom{m}{2},
$$

where, for $B=0, 1, 2, 3, 4, 5$ we have $\alpha=1, 1, 2, 3, 4, 5, \beta=4, 5, 4, 3, 2, 1, \gamma=1, 0, 0, 0, 0, 0$, respectively.

(2) Similarly, we compute $D(n; 1, 2, 4, 10)$, used p. 108. We have

$$
D_{1, 2, 4, 10} = x_1 x_2 x_4 x_{10} = (1+t) x_1^2 x_4 x_{10} = (1+t) (1+t^2)^2 x_3 x_{10} = (1+t) (1+t^2)^2 (1+t^4 + t^8 + t^{12} + t^{16})^3 (1+t^{10}) (1-t^{20})^{-4} = (1+t)
$$

$$
(1+2t^2+4t^4+6t^6+9t^{10}+13t^{16}+18t^{20}+24t^{14}+31t^{16}+39t^{18}+45t^{20}+52t^{22}+57t^{24}+63t^{26}+67t^{28}+69t^{30}+69t^{32}+\cdots + 2t^{60}+t^{62})
$$

$$
	imes \sum \binom{m+3}{3} t^{20m}. \text{ Hence } D(20m+2b+(0 or 1); 1, 2, 4, 10) = a\binom{m+3}{3} + b\binom{m+2}{3} + c\binom{m+1}{3} + d\binom{m}{3},
$$

where, for $b=0, 1, 2, \ldots, 9$ we have: $\alpha=1, 2, 4, 6, 9, 13, 18, 24, 31, 39, \beta=45, 52, 57, 63, 67, 69, 67, 63, 57, \gamma=52, 45, 39, 31, 24, 18, 13, 9, 6, 4, \delta=2, 1, 0, 0, \ldots, 0, \rho=0, \ldots, 0$, respectively.

Let us now give a more precise version of Theorem B (p. 110):

**Theorem C.** Supposing each pair $(a_i, a_j)$ relatively prime, we have:

$$
D(n; a_1, a_2, a_3, \ldots, a_k) := D(n) = \sum_{j=0}^{k-1} d_j n^j + V_{a_1}(n) + \cdots + V_{a_k}(n),
$$

where each $V_{a_j}(n)$ is a pcr of period $a_j$, $j=1, 2, \ldots, k$. So, $D(n)$ is a polynomial of degree $k-1$ in $n$, plus a sequence $V_A(n) := V_{a_1}(n) + \cdots + V_{a_k}(n)$, with period $A = \text{LCM}(a_1, a_2, \ldots, a_k)$. Moreover, denoting $S_i := a_{i+1} + a_{i+2} + a_{i+k} + \cdots$, $S_1 = a_2 + a_3 + a_4 + \cdots$, $P = a_1 a_2 a_3 \cdots a_k$, the following formulas hold:

$$
\begin{align*}
\delta_{k-1} &= \frac{1}{(k-1)!P}, \\
\delta_{k-2} &= \frac{S_1}{2(k-2)!P}, \\
\delta_{k-3} &= \frac{3S_1^2 - S_2}{24(k-3)!P}, \\
\delta_{k-4} &= \frac{S_1^2 - S_1 S_2}{48(k-4)!P},
\end{align*}
$$
Let us write \( \Psi_i = \Psi_i(t) \) for any polynomial whose degree is \( \leq i \). The theory of fractional decomposition implies:

\[
\mathcal{D}(t) := \sum_{n=0}^{\infty} D(n) t^n = \frac{1}{(1-t^3)(1-t^4)(1-t^5)} = \frac{1}{(1-t)^3 (1+t+t^2+\ldots+t^{a_1-1}) (1+t+t^2+\ldots+t^{a_2-1}) \ldots} = \frac{\Psi_{a_1-1}}{(1-t)^3} + \frac{\Psi_{a_2-1}}{1-t^{a_1-1}} + \frac{\Psi_{a_2-1}}{1-t^{a_2-1}} + \ldots
\]

Therefore, we obtain \( [Sr] \), and the relations \( \Psi_{a_1-1} = \Psi_{a_2-1} = \ldots = 0 \) involving the numerators of the preceding line imply the first condition (concerning the sum of values which must be equal to 0, p. 109). The standard methods for determining \( q_{knl} \) give \( [Ss] \). (For many other explicit formulas, see [Glaisher, 1909], [Sylvester, 1882].)

As an example, let us calculate \( D(n; 3, 5, 7) = D(n) \). Here, \( S_1 = 3 + 5 + \ldots + 15 \), \( S_2 = 3^2 + 5^2 + \ldots + 7^2 - 83, \) \( P = 3 \cdot 5 \cdot 7 = 105 \).

So, with \( [br, s] \):

\[
D(n) = \Psi_{a_1-1} + \Psi_{a_2-1} + \Psi_{a_3-1} + \ldots + \Psi_{a_n-1} \]

where \( \Psi_{a_1-1} \) abbreviates \( (x_1, x_2, x_3) \) \( \text{per} 3, \ldots \) \( \text{per} 3, \ldots \). Now, it is easy to compute \( D(0), D(1), D(2), \ldots, D(11) = 1, 0, 0, 1, 0, 1, 1, 1, 1, 2, 1 \) by carrying out \((1-t)^3 (1-t^2) (1-t^7) (1-t^9) \times (1+t+t^2+t^6+t^8) \times \ldots \) up to degree 11 or by using the recurrence \( D(n) = D(n-3) + D(n-5) + D(n-7) - D(n-8) - D(n-10) - D(n-12) + \ldots + D(n-15) \). If we insert these values of \( D(n) \) in \( [br, s] \), we must solve the following linear system of 15 equations with unknowns \( x_1, x_2, x_3, \ldots, x_{15} \) (the three last ones are the per condition, p. 109):

\[
x_1 + x_4 + x_9 = -35/315, \ x_1 + x_4 + x_{14} = -83/315, \ x_4 + x_7 + x_9 = 181/315, \ x_2 + x_4 + x_{13} = -188/315, \ x_1 + x_2 + x_3 = x_4 + x_5 + \ldots + x_{15} = 0.
\]

Solving this linear system, we find: \( (x_1, x_2, \ldots, x_{15}) \).

The use of a sum of 3 Cayley's per requires only 3 + 5 + 7 = 12 unknowns to find, whereas the use of one Herschel's cr would require 105 unknowns, this number being the length 3.5.7 of the oscillating term in \( D(n) \).

### PARTITIONS OF INTEGERS

#### SUPPLEMENT AND EXERCISES

1. **Recurrence relation for \( P(n, m) \)**. If \( P(n, m) \) stands for the number of partitions of the integer \( n \) into \( m \) summands (p. 94 and table p. 307), show that

\[
P(n, m) = P(n-1, m-1) + P(n-m, m),
\]

and that, for \( m > n/2 \),

\[
P(n, m) = \text{odd}(n).
\]

**[Hint:** Distinguish, in \( [lb, c] \), p. 95, the solutions with \( x_1 = 0 \) from those with \( x_1 \geq 1 \).]

2. **Recurrence relation for \( Q(n, m) \)**. As in the preceding exercise, prove that the number \( Q(n, m) \) of partitions of the integer \( n \) into \( m \) different summands satisfies: \( Q(n, m) = Q(n-m, m) - Q(n-m-1, m-1) \). Hence the first values of \( Q(n, m) \) and \( q(n) = \sum Q(n, m) \):

| \( m \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| \( q(n) \) | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |

3. **Convexity of \( p(n) \)**. The number \( p^*(n) \) of partitions of \( n \) into summands all \( \geq 1 \) equals \( p(n) - p(n-1) \) and this is an increasing function of \( n \). Deduce that the sequence \( p(n) \) (the number of partitions of \( n \)) is convex, in other words, that \( \Delta^2 p(n) = p(n-1) + p(n) \geq 0 \). More generally, \( \Delta^k p(n) \geq 0 \) for all \( k \geq 1 \).
4. *Some values of* $P(n, m)$ *and* $Q(n, m)$. For shortness, we write the sequence $(d_0, d_1, \ldots, d_{r-1})$ or $T_r$ of p. 109 as $[d_0, d_1, \ldots, d_{r-1}]$. $P(n, m)$ (or $Q(n, m)$) is the number of partitions of $n$ into $m$ arbitrary (or unequal) summands (see p. 99). Use $P(n, m) = Q(n + \binom{m}{2}, m)$ (which can be proved combinatorially), and hence $Q(n, m) - P(n - \binom{m}{2}, m)$ to show:

- **$P(n, 2) = \frac{1}{4} (2n - 1 + [1, -1])$**
- **$Q(n, 2) = \frac{1}{4} (2n - 3 - [1, -1])$**
- **$P(n, 3) = \frac{1}{72} (6n^2 - 7 - 9 [1, -1] + 8 [2, -1, -1])$**
- **$Q(n, 3) = \frac{1}{72} (6n^2 - 36n + 47 + 9 [1, -1] + 8 [2, -1, -1])$**
- **$P(n, 4) = \frac{1}{288} (2n^3 + 6n^2 - 9n - 13 + (9n + 9) \times [1, -1] - 32 [1, -1, 0] + 36 [1, 0, -1, 0])$**
- **$Q(n, 4) = \frac{1}{288} (2n^3 - 30n^2 + 135n - 175 + (9n - 45) \times [1, -1] - 32 [1, -1, 0] - 36 [1, 0, -1, 0])$.**

*5. Upper and lower bounds for* $P(n, m)$. Show that $P(n, m)$ and $Q(n, m)$, as defined on p. 94 and 99, satisfy:

$$Q(n, m) \leq \frac{1}{m!} \binom{n - 1}{m - 1} \leq P(n, m).$$

Use the fact that $Q(n, m) = p(n - \binom{m+1}{2}, m)$, [5] p. 105, to prove that

$$P(n, m) \leq \frac{1}{m!} \binom{n + \binom{m+1}{2}}{m - 1}$$

and

$$P(n, m) \sim \frac{1}{m!} \binom{n - 1}{m - 1}$$

for $n \to \infty$ and $m = O(n^{1/3})$. ([Erdős, Lehmer, 1941], [Gupta, 1947], [Rieger, 1959], [Wright, 1961].)

6. *The size of the smallest summand is given.* Let $a(n, m)$ be the number of partitions of $n$ such that the smallest summand equals $m$. Then:

$$\sum_{n \geq 0} a(n, m) t^n = t^m (1 - t^m) (1 - t^{m+1}) \ldots^{-1},$$

and

$$a(n, m) = a(n - m, m) + a(n + 1, m + 1),$$

where

$$a(n, n) = 1, \quad a(n, 1) = p(n - 1).$$

7. **Odd summands.** Let $p_1(n)$ be the number of partitions of $n$ into summands which are all odd, then we have $\sum_{n \geq 0} p_1(n) t^n = \left(1 - t \right) \left(1 - t^3 \right) \ldots^{-1}$, and $p_1(n) = q(n)$ (the number of partitions into unequal summands, p. 99). Prove this by formal methods and by combinatorial methods.

8. **The summands are bounded in number and size.** Let $p(n, m)$ be the number of partitions of $n$ into at most $m$ summands all $\leq l$. Show that:

$$A(t, u) := \sum_{n, m} p(n, m) t^n u^m = \prod_{l=0}^{l-1} (1 - ut^l)^{-1}.$$ 

Use a method analogous to that on p. 98 to show that:

$$A(t, u) = 1 + \sum_{m \geq 1} \frac{(1 - t^{l+1})(1 - t^{l+2}) \ldots (1 - t^{l+m})}{(1 - t)(1 - t^2) \ldots (1 - t^m)} t^m.$$ 

Deduce:

$$\sum_{n \geq 0} p(n, m, \leq l) t^n = \frac{(1 - t^{l+1})(1 - t^{l+2}) \ldots (1 - t^{l+m})}{(1 - t)(1 - t^2) \ldots (1 - t^l)}. $$

9. **The factorial number system.** For all $m \geq 1$ we have:

$$(1 + t)(1 + t^2 + t^2^2) \ldots (1 + t^m + t^2^{m+1} + \ldots + t^{m!-1}) = 1 + t + t^2 + \ldots + t^{m!-1}. $$

[Hint: This is equivalent to $1.1! + 2.2! + \ldots + n.n! = (n+1)! - 1$, which can be proved either by induction or by a combinatorial interpretation.] Use this to prove:

$$\prod_{j \geq 1} \sum_{0 \leq k \leq j} t^{j^k} = (1 - t)^{-1}, $$

and, for every integer $x \geq 0$, the existence of a unique sequence of integers $x_i$ such that

$$x = x_1 \cdot 1! + x_2 \cdot 2! + \ldots,$$

where $0 < x_i < i, i = 1, 2, 3, \ldots$. (See also Exercise 4, p. 255.)
10. With the binary number system. (1) For all \( m \geq 1 \), we have:
\[
(1 + ut)(1 + ut^2) \cdots (1 + ut^{2^m-1}) = \sum_{n=0}^{2^m-1} u^{D(n)} t^n.
\]
Here, \( D(n) \) stands for the number of ones in the binary (=base 2) representation of \( n \). Consequently (generalization in [Ostrowski, 1929]):
\[
\prod_{k=0}^{m} (1 + ut^k) = \sum_{n=0}^{2^m} u^{D(n)} t^n.
\]
(2) Also prove \( t(1-t)^{-1} = \sum_{k \geq 0} 2^k t^k (1 + t^2)^{-1} \) ([Teixeira, 1904]).

11. \( q \)-binomial coefficients. Let \( 0 < q < 1 \). We introduce
\[
\binom{x}{k}_q = \frac{(1 - q)x)(1 - q^{x-1}) \cdots (1 - q^{x-k+1})}{(1 - q^k)^k}
\]
\[
\langle x \rangle_q = \frac{1}{1 - q^x}, (\langle k \rangle)_q = 1.
\]
The \( q \)-binomial coefficients are defined by
\[
\binom{x+k}{k}_q = \binom{x}{k}_q q^k + \binom{x}{k-1}_q q^{k-1} + \cdots + q^{x-k+1}.
\]
They tend to the ordinary binomial coefficients when \( q \to 1 \).

(1) We have \( \binom{x}{k} = \binom{x}{k}_q \) for \( q = 1 \).

(2) \( \prod_{k=0}^{m-1} (1 + xt^k) = \sum_{k=0}^{m-1} q^{\binom{k}{2}} \binom{n}{k} t^k \).

(Observe the analogies with the expansions of \((1 + x)^n\) and \((1 - x)^{-n}\).)

For \( n \to \infty \), we recover [5k] (p. 105) and [5b] (p. 103). (3) \( b_n - \sum_{k=0}^{n} \)

\[
\binom{n}{k} a_k = \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \binom{n}{k} b_k.\]

(Compare [6a, e], p. 143.) (This is a very large subject, and we only touch upon it. For a completely updated presentation, see [Goldman, Rota, 1970].)

12. Prime numbers. To every integer \( n \geq 1 \), \( n = \sum_{p \geq 2} p^{a_p} \) as prime factor decomposition, we associate the number \( \omega(n) := a_2 + a_3 + \cdots, \omega(1) = 0 \). Thus, \( \omega(3500) = \omega(2^5 \cdot 5^3 \cdot 7) = 2 + 3 + 1 = 6 \). Then, for all complex numbers \( s \) and \( t \), such that \( \Re s > 1 \), and \( |t| \leq 1 \), the following equality between functions of \( s \) and \( t \) holds:
\[
\prod_{p \geq 2} (1 - t^p)^{-1} = \sum_{n \geq 0} \frac{\omega(n)}{n^s}.
\]

Here, in the infinite product, \( p \) runs through the set of all prime numbers (for \( t = 1 \), this is the famous factorization of the Riemann zeta function \( \zeta(s) = \sum_{n \geq 2} \frac{1}{n^s} \). See also Exercise 16, p. 162).

13. Durfee square identity for \( \sum p(n) t^n \). Prove the identity:
\[
\frac{1}{(1-t)(1-t^3)(1-t^5) \cdots} = 1 + \frac{t}{(1-t^3)^3} + \frac{t^4}{(1-t)^5} \frac{t^4}{(1-t)^2} + \frac{t^9}{(1-t)^7} (1-t)^2 (1-t)^2 (1-t)^2 + \cdots
\]

[Hint: Put \( \Phi(t,u) := \prod (1 - tu) (1 - t^{2u}) \cdots \) and observe that \( \Phi(t,u) = (1-tu) \Phi(t,u) \) and \( \Phi(t,u) = (1-tu) \sum_{n \geq 1} F_n(t,u) \) where \( F_n(t,u) = (1-tu) F_{n+1}(t,u) \) and \( \Phi(t) = \sum_{n \geq 1} C_n(t) \).

14. Some applications of the Jacobi identity. If we replace \( t \) by \( t^k \) and \( u \) by \( \pm t^l \) in the Jacobi identity, [5n] (p. 106), \( k \) and \( l \) integers \( \geq 0 \), prove:
\[
\prod_{k \geq 0} \left( 1 + t^{2k+1} \right) \left( 1 + t^{2k+2} \right) = \sum_{n \geq 0} \frac{1}{n^2 + 1}
\]
\[
\prod_{k \geq 0} \left( 1 - t^{2k+1} \right) \left( 1 - t^{2k+3} \right) = \sum_{n \geq 0} (-1)^n \frac{1}{n^2 + 1}.
\]

(1) Use this to prove the Euler identity, [5g] (p. 104), by putting \( k = \frac{1}{2}, l = \frac{1}{2} \).
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(2) If $k = \frac{1}{2}$, $l = \frac{1}{2}$:
\[
\prod_{i>0} \left(1 - t^{i+1}\right)(1 - t^{i+2})(1 - t^{i+3}) = \sum_{n \in \mathbb{Z}} (-1)^n t^{(5n+3)/2}.
\]

(3) If $k = \frac{1}{2}$, $l = \frac{1}{2}$:
\[
\prod_{i>0} \left(1 - t^{i+1}\right)(1 - t^{i+3})(1 - t^{i+4}) = \sum_{n \in \mathbb{Z}} (-1)^n t^{(5n+1)/2}.
\]

(4) If $k = 1$, $l = 0$:
\[
\prod_{i>0} \left(1 - t^{2i+1}\right)(1 - t^{2i+2}) = \sum_{n \in \mathbb{Z}} (-1)^n t^2.
\]

15. Use of the function $\|x\|$, the integer closest to $x$. With the notation of [6f] (p. 110), we have, in addition to [6g, q]:
\[
D(n; 1, 2, 5) = \|n(n + 4)/20\|;
D(n; 1, 2, 7) = \|(n + 3)(n + 7)/28\|;
D(n; 1, 3, 5) = \|n(n + 3)(n + 6)/30\|;
D(n; 1, 3, 7) = \||(n + 3)(n + 8)/42\|;
D(n; 1, 5, 7) = \|n^2 + 13n + 36)/60\|;
D(n; 1, 2, 3, 5) = \|(n + 3)(2n + 9)(n + 9)/360\| = \|(n + 2)(n + 8)(2n + 13)/360\|;
D(n; 1, 2, 3, 7) = \|(n + 3)(2n + 9)(n + 9)/360\| = \|(n + 2)(n + 8)(2n + 13)/360\|;
P(n, 2) = Q(n + 1, 2) = \|(2n - 1)/14\|;
P(n, 3) = Q(n + 3, 3) = \|n^2/12\|;
P(n, 4) = Q(n + 6, 4) = \|n^2(n + 3)/144\| for $n$ even; and
\[
\|n - 1\|^{2}(n + 1)/144\| for $n$ odd.
\]

(For plenty of other such formulas, see [Popoviciu, 1953]).

16. Infinite power series as an infinite product. To any sequence $(a_1, a_2, a_3, \ldots)$, let us associate $(b_1, b_2, b_3, \ldots)$ such that
\[
f(t) = 1 + \sum_{m \geq 1} a_m t^m = \prod_{n \geq 1} (1 + b_n t).
\]

(1) We have $a_n = \sum b_1 b_2^2 b_3^3 \ldots$, where $e_1, e_2, e_3, \ldots = 0$ or 1, and $e_1 + 2e_2 + 3e_3 + \ldots = n$. So, $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3 + b_1 b_2$, $a_4 = b_4 + b_1 b_3$,

$\ldots$. Evidently, $b_1 = b_2 = \ldots = 1$ implies $a_n = q(n)$, the number of partitions of $n$ into unequal summands (p. 99). (2) Conversely, calculate $b_n$ as a polynomial in $a_1, a_2, \ldots$. So, $b_1 = a_1, b_2 = a_2$, $b_3 = a_3 - a_2 a_1$, $b_4 = a_4 - a_2 a_1 + a_2^2$, $b_5 = a_5 - (a_4 a_1 + a_3 a_2) + (a_3 a_1^2 + a_1 a_2^2) - a_1^3 a_2$, $b_6 = a_6 - (a_4 a_1 + a_3 a_2) + (a_3 a_1^2 + a_1 a_2^2) - a_1^3 a_2$, $a_7 = a_7 - (a_6 a_1 + a_5 a_2 - a_2 a_3) + (a_5 a_1 + 2a_4 a_2 - a_2 a_3 - a_2^2) - (a_4 a_1^2 + 3a_3 a_2^2 - a_2 a_3 - a_2^2) + (a_3 a_1^3 - 2a_2 a_1 a_2) - a_1 a_2^3 + a_2^3 - a_2^4 - a_2^5 - \ldots$ If $a_1 = a_2 = \ldots = 1$, then $b_n = 0$, except $b_{2n} = 1$. (3) When $f(t) = e^{-t}$, prove the following property: $(b_n = l/n) \Leftrightarrow (n$ is prime) ([Kolberg, 1960]).

17. Three summations of denumerants. Verify the following summation formulas ([Pólya, Szegő, I, 1926, p. 3, Exercises 22, 23, 24]): \[\sum_{i>0} D(n - i; 1 + i, 2 + i) = n + 1; \sum_{i>0} D(n - 2i - 1; 1 + i, 2 + i) = n + 2 - d(n),\]
where $d(n)$ is the number of divisors of $n$. [Hint: Use Exercise 16, p. 162.]

18. Integer points. (1) The number of points $(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$, with integer coordinates, $x_i \in \mathbb{Z}$, such that $|x_1| + |x_2| + \ldots + |x_n| \leq p$, integer $\geq 0$, equals: \[\sum_{i=0}^{n-1} 2^{n-i} \binom{n}{i} \binom{p}{n-i} = (\ast)\] ([Pólya, Szegő, I, 1926], p. 4, Exercise 29). (2) The number of solutions with integers $x_i \geq 1$, $i \in [n]$, that satisfy $1 \leq x_1 < x_2 < \ldots < x_n$, $x_1 < k+1$, $x_2 < k+2$, \ldots, $x_n < k+n$, equals \[\binom{k+2n}{n} (k+1)/(k+n+1).\] ([Whitworth, 1901], p. 115–16; [Barbenson, 1965], [Carlitz, Roselle, Scoville, 1971]).

*19. Rational points in a polyhedron ([Ehrhart, 1967]). We denote the set of points in $\mathbb{R}^d$ whose coordinates are multiples of $1/n$ by $G_n^d$. The problem of the denumerants ([6b] p. 109), which can also be written $a_1 (x_1/n) + a_2 (x_2/n) + \ldots + a_n (x_n/n) = 1$, is hence equivalent to finding the number $Z(n)$ of points of $G_n^d$ lying in the hyperplane defined by $a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = 1$, whose $k$ vertices are the points $A_1 = (1/a_1, 0, 0, \ldots), A_2 = (0, 1/a_2, 0, \ldots), \ldots$. More generally, let $P$ be a polyhedral region of $\mathbb{R}^d$, whose vertices are $A_1, A_2, \ldots, A_n$, with rational coordinates; each face may or may not belong to $P$. For each vertex $A_i$, let $a_i$ be the LCM of the denominators of $A_i$. Then we denote the number of points in $P \cap G_n^d$ by $I(n)$; we put $I(0) := 1$. (1) There
exists a polynomial \( P(t) \) of degree less than \( \sum a_i \) such that

\[
\mathcal{J}(t) := \frac{\sum I(n) t^n}{1 - \prod (1 - t^{a_i})} = \frac{P(t)}{\prod (1 - t^{a_i})},
\]

\[ [\text{Hint: First treat the case of a simplex.}] \]

For example, if \( \mathcal{S} \) is the open polygon in \( \mathbb{R}^2 \) whose vertices are \( A_1 = (0, 0), A_2 = (1, 0), A_3 = (\frac{3}{4}, \frac{3}{4}), A_4 = (0, 1) \), we have \( a_1 = a_2 = a_3 = 1, a_4 = 6 \). Hence \( \sum_{n \geq 0} I(n) t^n = \frac{P(t)}{(1 - t)^2(1 - t^6)^{\frac{1}{2}}} \), \( \deg P \leq 8 \). (2) The rational fraction \( \mathcal{S}(t) \) can be simplified so that the exponent of the factor \( (1 - t) \) in the denominator is \( \leq d + 1 \). For the preceding example we then get \( \mathcal{S}(t) = P(t)(1 - t)^{-2} \times (1 - t^6)^{-\frac{1}{2}} \) and \( P(t) \) can be determined by \( I(0), I(1), I(2), \ldots, I(7) = 1, 0, 1, 2, 6, 9, 13, 18 \), respectively, which we obtain by direct inspection. Hence \( P(t) = 1 - 2t + t^2 + t^3 + t^4 + t^5 + 3t^6 \) and \( \frac{1}{(1 - t)^2}(1 - t^6)^{-\frac{1}{2}} \) can be determined by \( Z(0), Z(1), Z(2), \ldots, Z(7) = 1, 0, 0, 1, 3, 6, 9, 13, 18 \), respectively, which we obtain by direct inspection. Hence \( \mathcal{S}(t) = P(t)(1 - t)^{-2} \times (1 - t^6)^{-\frac{1}{2}} \) and \( P(t) \) can be determined by \( I(0), I(1), I(2), \ldots, I(7) = 1, 0, 1, 2, 6, 9, 13, 18 \), respectively, which we obtain by direct inspection. Hence \( P(t) = 1 - 2t + t^2 + t^3 + t^4 + t^5 + 3t^6 \) and \( \frac{1}{(1 - t)^2}(1 - t^6)^{-\frac{1}{2}} \) can be determined by \( Z(0), Z(1), Z(2), \ldots, Z(7) = 1, 0, 0, 1, 3, 6, 9, 13, 18 \), respectively, which we obtain by direct inspection. Hence \( \mathcal{S}(t) = P(t)(1 - t)^{-2} \times (1 - t^6)^{-\frac{1}{2}} \) and \( P(t) \) can be determined by \( I(0), I(1), I(2), \ldots, I(7) = 1, 0, 1, 2, 6, 9, 13, 18 \), respectively, which we obtain by direct inspection. Hence \( P(t) = 1 - 2t + t^2 + t^3 + t^4 + t^5 + 3t^6 \) and \( \frac{1}{(1 - t)^2}(1 - t^6)^{-\frac{1}{2}} \) can be determined by \( Z(0), Z(1), Z(2), \ldots, Z(7) = 1, 0, 0, 1, 3, 6, 9, 13, 18 \), respectively, which we obtain by direct inspection. Hence \( \mathcal{S}(t) = P(t)(1 - t)^{-2} \times (1 - t^6)^{-\frac{1}{2}} \) and \( P(t) \) can be determined by \( I(0), I(1), I(2), \ldots, I(7) = 1, 0, 1, 2, 6, 9, 13, 18 \), respectively, which we obtain by direct inspection. Hence \( P(t) = 1 - 2t + t^2 + t^3 + t^4 + t^5 + 3t^6 \) and \( \frac{1}{(1 - t)^2}(1 - t^6)^{-\frac{1}{2}} \) can be determined by \( Z(0), Z(1), Z(2), \ldots, Z(7) = 1, 0, 0, 1, 3, 6, 9, 13, 18 \), respectively, which we obtain by direct inspection. Hence \( \mathcal{S}(t) = P(t)(1 - t)^{-2} \times (1 - t^6)^{-\frac{1}{2}} \) and \( P(t) \) can be determined by \( I(0), I(1), I(2), \ldots, I(7) = 1, 0, 1, 2, 6, 9, 13, 18 \), respectively, which we obtain by direct inspection. Hence \( P(t) = 1 - 2t + t^2 + t^3 + t^4 + t^5 + 3t^6 \) and \( \frac{1}{(1 - t)^2}(1 - t^6)^{-\frac{1}{2}} \) can be determined by \( Z(0), Z(1), Z(2), \ldots, Z(7) = 1, 0, 0, 1, 3, 6, 9, 13, 18 \), respectively, which we obtain by direct inspection. Hence \( P(t) = 1 - 2t + t^2 + t^3 + t^4 + t^5 + 3t^6 \) and \( \frac{1}{(1 - t)^2}(1 - t^6)^{-\frac{1}{2}} \) can be determined by \( Z(0), Z(1), Z(2), \ldots, Z(7) = 1, 0, 0, 1, 3, 6, 9, 13, 18 \), respectively, which we obtain by direct inspection.

\[ \[21. \text{The number of score vectors of a tournament.} \text{ (Defined on p. 68. See [Bent, Narayana, 1964] and [*Moon, 1968, p. 66.) We want to determine the number of solutions with integers } s_i \text{ of:} \]

\[ [\alpha] \quad s_1 < s_2 < \cdots < s_n < n - 1 \]

\[ [\beta] \quad s_1 + s_2 + \cdots + s_n > \binom{k}{2}, \quad k \in [n - 1] \]

\[ [\gamma] \quad s_1 + s_2 + \cdots + s_n = \binom{n}{2}. \]

Let \([t, l]^n \) be the number of solutions of \([x, \beta, \delta] \):

\[ [\delta] \quad s_1 + s_2 + \cdots + s_n = l, \quad s_n = t. \]

Hence \([t, l]^n = 1 \) for \( t = l \) and \( = 0 \) if not. (1) We have \([t, l]^n = \sum_{k \leq t} [l, l - 1]^{-k} \). (2) Hence \( s(n) = \sum [l, \binom{n}{2}] \). (3) Compute from this the first few values. (There is no exact formula for \( s(n) \) and there is a conjecture that the ratio \( s(n+1)/s(n) \) increases towards 4.)

\[ \[22. \text{Relatively prime summands.} \text{ The number } R_k(n) \text{ of integer solutions } x_1, x_2, \ldots, x_k = n \text{ such that these integers are relatively prime, is such that ([Gould, 1964a]. See also Exercise 16 (5), p. 161):} \]

\[ \sum_{x_i \geq 1} R_k(n) \frac{t^k}{\prod (1 - t^{x_i})} = \frac{t^k}{\prod (1 - t^{x_i})}. \]

\[ \[23. \text{Compositions.} \text{ (1) A composition of the integer } n \text{ into } m \text{ summands, or } m \text{-composition, is any solution } x = (x_1, x_2, \ldots, x_m) \text{ of } x_1 + x_2 + \cdots + x_m = n \text{ with integer } x_i \geq 1, i \in [m] \text{ (the order of the summands counts); } C_m(n) \text{ stands for the set of } m \text{-compositions of } n. \text{ Show that } C(n, m) = |C_m(n)| - \]
has the following GF: \( \sum_{n} C(n, m)t^n = \left(1 + t\right)^{m+1} \).

More generally, the number \( C(n, m; A) \) of solutions of \( \sum_{i} x_i = n \), where for all \( i \in \{m\} \), \( x_i \in A = \{a_1, a_2, a_3, \ldots\} \) with \( 1 < a_1 < a_2 < \cdots \), is such that:

\[
1 + \sum_{n \geq m} C(n, m; A)t^n = \left[ 1 - u (e^t + e^{2t} + \cdots) \right]^{-1}.
\]

In how many ways can one put stamps to a total value of 30 cents on an envelope, if one has stamps of 5, 10 and 20 cents, which are glued in a single row onto the envelope (so the order of the stamps counts!)?

[Answer: 18.]

More generally, for 5n cents (instead of 30, where \( n = 6 \))
and using notation \([65]\) on p. 110, the number of ways becomes: \( \left[0,690367\ldots(1,754878\ldots)^{\infty}\right]! \).

(3) Returning to (1), we endow \( \mathcal{C}_m(n) \) with an order relation by putting, for \( x = (x_1, x_2, \ldots, x_m) \) and \( x' = (x_1', x_2', \ldots, x_m') \):

\[
x < x' \iff \forall k \in \{m\}, \quad \sum_{i=1}^k x_i < \sum_{i=1}^k x_i'.
\]

Show that \( \mathcal{C}_m(n) \) becomes a distributive lattice in this way. (4) For each \( x \in \mathcal{C}_m(n) \) let \( \xi := \{ v \mid v \in \mathcal{C}_m(n), \ v < x \} \), then \( \sum_{x \in \mathcal{C}_m(n)} |\xi| = (n! / (n-1)!^m) \) ([Narayana, 1955]).

24. Denumerants with multi-indexes. For vectors \( n = (n_1, n_2, \ldots, n_k) \) (or multi-indexes, p. 36), a partition theory can be developed analogous to that given in this chapter. See for instance [*MacMahon, 11, 1916], p. 54 and [Blakley, 1964a]. Let \( \mathcal{S} \) be the system of \( k \) equations:

\[
a_{i_1} x_1 + a_{i_2} x_2 + \cdots + a_{i_k} x_k = n \quad \text{for } i \in \{k\},
\]

where the \( a_{i,j} \) are integers such that \( 1 < a_{i,1} < a_{i,2} < a_{i,3} < \cdots \). Show that the number \( D((n); (a)) \) of solutions of \( \mathcal{S} \) in integers \( x_i \geq 0 \) has GF:

\[
\sum_{x_1, x_2, \ldots, x_k \geq 0} D((n), (a)) t_1^{x_1} t_2^{x_2} \cdots t_k^{x_k} = \prod_{j=1}^{k} \left[ 1 - \frac{t_j^n}{1 - \frac{t_j^n}{1 - \frac{t_j^n}{\cdots}}} \right]^{-1}.
\]

Let \( Q(n, r) \) be the number of arrays (or matrices) of integers \( a_{i,j} \geq 0 \), \( 1 \leq i, j \leq n \), such that \( \sum_{i=1}^n a_{i,j} = \sum_{j=1}^n a_{i,j} = r \), for all \( i, j \).

(1) \( Q(1, r) = 1, \ Q(2, r) = r + 1, \ Q(3, r) = \binom{r+2}{2} + 3 \binom{r+3}{4} \).

(2) \( Q(4, r) = \binom{r+3}{3} + 20 \binom{r+4}{5} + 152 \binom{r+5}{7} + 352 \binom{r+6}{9} \).

More generally, \( Q(n, r) \) is a polynomial with degree \( (n-1)^2 \) with respect to \( r \).

*25. Counting magic squares. Let \( \varphi(x) := \sum_{k \geq 0} (3k)! (k!)^{-2} x^k \); then \( \sum_{n \geq 0} a_n t^n (n!)^{-2} = e^{\varphi(x) / 2} (1 - t)^{-1/2} \) and \( a_n = \frac{n}{n!} a_{n-1} - \frac{1}{n-1} j (n-1) \) \( a_{n-2} \).

Moreover, \( a_n = n! \binom{2n}{n} / (n-1)! \), where the \( A_n \) are integers.


\begin{align*}
\begin{array}{cccccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
 A_n & 1 & 1 & 3 & 7 & 47 & 207 & 2249 & 14501 \\
\end{array}
\end{align*}

Let \( b_n = Q(n, 3) \) and \( \varphi(x) := \sum_{k \geq 0} (3k)! (k!)^{-2} x^k \); then \( \sum_{n \geq 0} b_n t^n (n!)^{-2} = e^{\varphi(x) / 2} (1 - t)^{-1/2} \) and \( b_n = \frac{n}{n!} a_{n-1} - \frac{1}{n-1} j (n-1) \) \( a_{n-2} \).

Moreover, \( b_n = n! \binom{2n}{n} / (n-1)! \), where the \( A_n \) are integers.

\begin{align*}
\begin{array}{cccccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
 b_n & 1 & 1 & 4 & 55 & 2008 & 133040 & 602857300 & 157900624000 \\
\end{array}
\end{align*}

26. Standard tableaux. Each Ferrers diagram representing a certain partition of \( n \) can be considered in the obvious way as a 'descending wall' \( M \), or 'profile'. Figure 29 represents the wall associated with the diagram of Figure 25 (p. 100). The 'stone' \( (i, j) \) is the one with 'abscissa' \( i \) and 'ordinate' \( j \). We are interested in the number \( v(M) \) of different ways in which \( M \) can be built up by piling stones one by one on top of each other, in such a way that at every stage the already constructed part is a 'descending wall'. Figure 30 gives a permissible numbering of the stones, thereby defining a so-called 'standard' tableau, also called Young tableau. For a given wall \( M \) we write on each stone \( (i, j) \) the number of stones situated above and to the right of it, itself included. The table of numbers \( z(i, j) \), obtained in this way, is represented in Figure 31. Hence the number of standard tableaux \( v(M) \), equals \( n! (\prod_{(i, j) \in \mathcal{Z}(i, j)})^{-1} \). We refer to [Kreweras, 1965, 1966a, b, 1967] for a study and a very complete bibliography of the problem, as well as for a
Fig. 29. Fig. 30. Fig. 31.

generalization to the case that part of the wall, say $M'$, already exists, that is, it will be incorporated into $M$. See also [Berge, 1968], pp. 49–59. We remark that the generalization to higher dimensions, in the sense of p. 103, is still an open problem.

27. Perfect partitions. A perfect partition of an integer $n \geq 1$, is one that 'contains' precisely one partition of each integer less than $n$. In other words, if we consider the partition as a solution of $x_1 + 2x_2 + \cdots = n$, we call it perfect if for each integer $1 \leq i \leq n$ there exists a single solution of $t_1 + 2t_2 + \cdots = i$, where $0 \leq t_i \leq x_i$, $i = 1, 2, \ldots$. So a perfect partition represents a set of weights such that each weight of $i$ grams, $1 \leq i \leq n$, can be realized in exactly one way.

Show that the number of perfect partitions of $n$ equals the number of ordered factorisations of $n+1$, omitting unit factors. Thus, for $n = 7$, we have $8 = 4 \cdot 2 = 2 \cdot 4 = 2 \cdot 2 \cdot 2$, hence there are 4 perfect partitions, $1^7$, $1^3 4$, $1^2 2^3$, $1^2 2^2 2$.

28. Sums of multinomial coefficients. Let us write $A(n)$ for the sum of the multinomial coefficients which occur in the expansion of $(x_1 + x_2 + \cdots + x_n)^n$. For example since $(x_1 + x_2 + \cdots + x_n)^3 = \sum x_i^3 + 3 \sum x_i^2 x_j + 6 \sum x_i x_j x_k$ (see p. 29) we have $A(3) = 1 + 3 + 6 = 10$. Prove that

$$\sum_{x \geq 0} A(n) \frac{t^n}{n!} = \frac{1}{(1 - \frac{t}{1!})(1 - \frac{t^2}{2!})(1 - \frac{t^3}{3!}) \cdots}$$

and study other properties of these numbers.

<table>
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<th>7</th>
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3.1. Expansion of a product of sums; Abel identity

The following notations slightly generalize the binomial and multinomial identities of pp. 12 and 28.

**Theorem A.** Let $\mathcal{R}$ be a relation between two finite sets $M$ and $N$ ($\mathcal{R} \subseteq M \times N$, $|M| = m$, $|N| = n$), Figure 32, and let $u(x, y)$ be a double sequence defined on $\mathcal{R}$ and with values in a ring $A$ (mostly $A = \mathbb{R}$ or $\mathbb{C}$). If $(x | \mathcal{R})$ stands for the first section (p. 59) of $\mathcal{R}$ by $x$, then we have:

$$[1a] \prod_{x \in M} \sum_{y \in (x | \mathcal{R})} u(x, y) = \sum_{\varphi \in N^{NM}} \prod_{x \in M} u(x, \varphi(x)).$$

The summation in the second member of $[1a]$ is taken over all maps $\varphi$ of $M$ into $N$, whose ' graphical representation' is a subset of $\mathcal{R}$.

---

Let us suppose that the projection of $\mathcal{R}$ onto $M$ is just equal to $M$,
because if not, then both members of \([1a]\) equal zero. We number the elements of \(M\) and \(N,\) \(M := \{x_1, x_2, \ldots, x_m\},\) \(N := \{y_1, y_2, \ldots, y_n\}.\) If \(\mathcal{R} = M \times N,\) then the first member of \([1a]\) can be written as \(\prod_{i=1}^{m} \sum_{j=1}^{n} u(x_i, y_j).\) This is a product of \(m\) sums: The choice of a term in each of the \(m\) factors gives one term of the expansion, and two different choices give rise to two differently written terms. Now, any such choice is just a map \(\phi\) from \(M\) into \(N;\) hence \([1a].\) If \(\mathcal{R} \neq M \times N,\) then \(u(x, y)\) can be extended to the whole of \(M \times N\) by defining \(u(x, y) := 0\) for \((x, y) \notin \mathcal{R}.\) Then we can apply the preceding result, observing that the \(\phi\) whose graph is not contained in \(\mathcal{R}\) give a contribution zero to the second member of \([1a].\)

Using \([1a],\) the binomial and multinomial identities can easily be recovered.

We now show a deep generalization of the binomial identity.

**Theorem B.** (Abel identity [Abel, 1826]). For all \(x, y, z\) we have:

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k. \tag{1b}
\]

(In a commutative ring, for instance. But \([1b]\) also can be considered as an identity in the ring of polynomials in three indeterminates \(x, y, z).\) For \(z=0\) we recover the binomial identity \([6a]\) (p. 12).

**First proof (Lucas).** We introduce the Abel polynomials

\[
a_k(x, z) := x(x-kz)^{k-1}/k! \quad \text{for} \quad k \geq 1, \quad a_0 := 1.
\]

We have, successively,

\[
\frac{\partial}{\partial x} a_k(x, z) = (x-kz)^{k-1} + (k-1)x(x-kz)^{k-2})/k! = a_{k-1}(x-z, z)
\]

\[
\frac{\partial^2}{\partial^2 x} a_k(x, z) = \frac{\partial}{\partial x} a_{k-1}(x-z, z) = a_{k-2}(x-2z, z)
\]

\[
\frac{\partial^l}{\partial^l x} a_k(x, z) = a_{k-j}(x-jz, z).
\]

Now, for fixed \(z,\) the \(a_k(x, z)\) form a basis of the set of polynomials in \(x,\) because their degree equals \(k(=0, 1, 2, \ldots).\) Hence, every polynomial

\[P(x)\] can be uniquely expressed in the form \(P(x) = \lambda_0 a_0 + \lambda_1 a_1 + \lambda_2 a_2 + \cdots,\) where the \(\lambda_j\) only depend on \(z.\) Now, with \([1d]\) for \((\ast)\):

\[
P^{(l)}(x) = \frac{d^l}{dx^l} P(x) = \sum_{k=0}^{n} \lambda_k \frac{\partial^l}{\partial x^l} a_k = \lambda_j + \lambda_{j+1} a_1(x-jz, z) + \cdots
\]

which gives \(\lambda_j = P^{(l)}(jz),\) by putting \(x=jz.\) So finally, for every polynomial \(P(x)\) we have:

\[
\begin{align*}
[1e] & \quad P(x) = \sum_{k=0}^{n} a_k(x, z) P^{(k)}(kz),
\end{align*}
\]

from which \([1b]\) follows by putting \(P(x) = (x+y)^n.\) 

We still observe that if we apply \([1e]\) to \(P(x) = \alpha_n(x+y, z),\) then we get the convolution

\[
[a] \quad \alpha_n(x+y, z) = \sum_{k=0}^{n} a_k(x, z) \alpha_{n-k}(y, z).
\]

See also [Hurwitz, 1902], [Jensen, 1902], [Kaucky, 1968], [*Riordan, 1968], p. 18–27, [Robertson, 1962], and [Salié, 1951], who gives a large bibliography.

**Second proof (Françon).** All the notions of p. 71 concerning the Foata coding of \([n]_n\) will be supposed known. Let \(E \subset [n+2]^{n+2}\) be the set of functions of \([n+2] := \{1, 2, \ldots, n, n+1, n+2\}\) such that elements \((n+1)\) and \((n+2)\) are fixed points. So, \(E_E = E \times E\) be the set of functions whose exycle containing the element \((n+1)\) has \(A_i := x + \{n + 1\}\) as set of nodes. Obviously, the factorization \(E(x) = E_1 E_2\) holds, where \(E_i\) is the set of acyclic functions acting on \(A_i\) with the root \((n+1)\) only, and \(E_2\) is the set of functions acting on \([n+2] \setminus A_i\) and having the element \((n+2)\) as a fixed point. Then

\[
E_{E(x)} = E_{E_1} E_{E_2} = t_{n+1} t_{n+2} (t_{n+1} + \sum_{i \in [n]} t_i)^{n-1}. \tag{1f}
\]

But we have the division \(E = \sum_{x \in [n]} E(x).\) Therefore, \(E_{E(x)} = \sum_{x \in [n]} E_{E(x)}\) in \(E\).

In other words, after cancelling \(t_{n+1} t_{n+2}:\)

\[
(t_1 + t_2 + \cdots + t_{n+2})^n = \sum_{x \in [n]} \left(\sum_{i \in [n]} t_i \right)^{n-1} x \times \left(\sum_{i \in [n]} t_i \right)^n.
\]
Now, put \( t_{n+1} = x, t_{n+2} = y - n, t_1 = t_2 = \cdots = t_n = -z \) to obtain (1b) after collecting the \( k \) such that \( |x| = k. \)

Of course, considering more than 2 fixed points, or other sets of functions, would give interesting other results (see Exercise 20, p. 163).

The following is an equivalent formulation of the Abel identity (1b), which generalizes (1a).

**Theorem C.** For any formal series (hence for each polynomial) \( f(t) \), we have:

\[
(1g) \quad f(t) = \sum_{k \geq 0} \frac{t(t - ku)^{k-1}}{k!} f^{(k)}(ku),
\]

where \( u \) is a new indeterminate, and \( f^{(k)} \) the \( k \)-th derivative of \( f \).

(For a study of the convergence of (1g), \( t, u \in \mathbb{C} \), see [Halphen, 1881, 1882], [Pincherle, 1904].)

For \( u = 0 \), we find back the ordinary (formal) Taylor formula.

### 3.2. Product of Formal Series; Leibniz Formula

The series used in this chapter will be always formal Taylor series. By definition, such a series is written as follows (for the meaning of the abbreviated notations \( x, k, \text{etc.} \), see p. 36):

\[
(2a) \quad f = f(t) = f(t_1, t_2, \ldots, t_k) = \sum_{x \in \mathbb{R}^k} f_x \frac{t^x}{x!} = \sum_{x_1, x_2, \ldots, x_k \geq 0} f_{x_1, x_2, \ldots, x_k} \frac{t_1^{x_1}}{x_1!} \frac{t_2^{x_2}}{x_2!} \cdots \frac{t_k^{x_k}}{x_k!}.
\]

The \( f_x \) are called Taylor coefficients of \( f \).

**Theorem A (Leibniz formula).** Let \( f \) and \( g \) be two formal series, with Taylor coefficients \( f_x, g_x, \lambda \in \mathbb{R}^k \), and let \( h \) be the product series, \( h = fg \). Then, the Taylor coefficients \( h_x \) of \( h \) can be expressed as follows:

\[
(2b) \quad h_x = h_{\mu_1, \mu_2, \ldots, \mu_k} = \sum_{x_1, x_2, \ldots, x_k \geq 0} \frac{\mu_1! \mu_2! \cdots \mu_k!}{x_1! x_2! \cdots x_k!} f_{x_1} f_{x_2} \cdots f_{x_k} \lambda_{x_1} \lambda_{x_2} \cdots \lambda_{x_k},
\]

where the summation takes place over all systems of integers \( x_1, x_2, \ldots, x_k, \lambda_1, \lambda_2, \ldots, \lambda_k \) such that \( x_1 + \lambda_1 = \mu_1, x_2 + \lambda_2 = \mu_2, \ldots, x_k + \lambda_k = \mu_k \). In other words:

\[
(2c) \quad h_{\mu_1, \ldots, \mu_k} = \sum_{x_1, \ldots, x_k} \left( \mu_1 \right)_{x_1} \cdots \left( \mu_k \right)_{x_k} f_{x_1} \cdots f_{x_k} \lambda_{x_1} \cdots \lambda_{x_k},
\]

or, in abbreviated notation:

\[
(2d) \quad h_x = \sum_{x_1 + x_2 + \cdots + x_k = x} \frac{\mu!}{x_1! x_2! \cdots x_k!} f_{x_1} f_{x_2} \cdots f_{x_k} \lambda_{x_1} \cdots \lambda_{x_k}.
\]

It suffices to apply definition (12g) (p. 37) of the product \( fg \).

Formula (2d) can immediately be generalized to a product \( h \) of \( r \) formal series \( f_1, f_2, \ldots, f_r \), \( h = \prod_{i=1}^r f_i \). So:

\[
(2e) \quad h_x = \sum_{x_1 + x_2 + \cdots + x_r = x} \frac{\mu!}{x_1! x_2! \cdots x_r!} f_{x_1} f_{x_2} \cdots f_{x_r} \lambda_{x_1} \cdots \lambda_{x_r},
\]

where the summation is extended over systems of multi-indices \( \lambda_{x_1} \in \mathbb{R}, \lambda_{x_2} \in \mathbb{R}, \ldots, \lambda_{x_r} \in \mathbb{R} \) such that:

\[
(2f) \quad \lambda_{x_1} + \lambda_{x_2} + \cdots + \lambda_{x_r} = \mu.
\]

We observe, by (2f) and Theorems B and D (p. 15), that the summation of (2e) contains \( \prod_{j=1}^r \binom{\mu_j + r - 1}{r - 1} \) terms which is the number of solutions of (2f).

Actually, the exact formula (2b) allows us to calculate effectively the (partial) derivatives of a product of two functions. For each function \( F(x) = F(x_1, x_2, \ldots, x_k) \) defined in a neighbourhood of \( d = (a_1, a_2, \ldots, a_k) \in \mathbb{R}^k \) and of class \( C^{\infty} \) in this point, and for any \( x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k \), we put:

\[
(2g) \quad f_x := \frac{\partial^{|x|} F}{\partial x_1^{x_1} \cdots \partial x_k^{x_k}} = \frac{\partial^{x_1} F(x_1, \ldots, x_k)}{\partial x_1^{x_1} \cdots \partial x_k^{x_k}}(x_1, \ldots, x_k) = (a_1, \ldots, a_k),
\]

\[
\frac{\partial F}{\partial x_0, \ldots, 0} := F(a_1, \ldots, a_k)
\]
and let:

\[ f := \tau_a(F) = \sum_{n \in \mathbb{K}} f_n \frac{t^n}{n!} \]

be the formal Taylor series associated with the function \( F \) in \( a \).

**Theorem B.** Let the two functions \( F \) and \( G \) be of class \( C^\infty \) in \( a \in \mathbb{R}^k \), and let \( H := F \cdot G \). Between the three formal series \([2i] \): \( f := \tau_a(F) \), \( g := \tau_a(G) \), \( h := \tau_a(H) \), there exists the relation \( h - fg \) in the sense of the product of formal series ([12g], p. 37).

This is a well-known property of functions of class \( C^\infty \) in a point. (See, for example, [*Valiron, I, 1958*, p. 235.)

**Theorem C.** Let \( r (\geq 2) \) functions \( F_{(i)} = F_{(i)}(x), i \in [r], x \in \mathbb{R}^k \), be given. All of class \( C^\infty \) in \( a \in \mathbb{R}^k \), and let \( f := \tau_a(F_{(i)}), i \in [r], f := \sum_{i \in [r]} f_{(i)} \).

\( f_{(i)}, i \in [r], f_{(i)} := \frac{\partial^r f}{\partial x_1 \partial x_2 \cdots \partial x_r} \), \( \lambda \langle i \rangle ! \) be their associated formal Taylor series (cf. [2h]). Then, the successive derivatives \( h_a \) of the function \( H := \prod_{i=1}^r F_{(i)} \) in \( a \) given by formula [2e] (and particularly by [2b, c, d] if \( r = 2 \)).

This is an immediate consequence of Theorems A and B.

In this way we recover for the product \( H(x) = F(x) \cdot G(x) \) of two functions of one variable the usual Leibniz formula:

\[ h_m = \sum_{l=0}^{m} \binom{m}{l} f_{l}g_{m-l}, \]

where

\[ f_{l} := \left. \frac{d^l F(x)}{dx^l} \right|_{x=a}, \]

etc., \( f_{0} := f(a) \). Similarly, for the product \( H(x) = F_{(1)}(x) \cdots F_{(r)}(x) \) of \( r \) functions we get:

\[ h_m = \prod_{l=1}^{r} \binom{m}{l} f_{l}g_{m-l}, \]

where:

\[ l \langle 1 \rangle + \cdots + l \langle r \rangle = m, f_{l} := \left. \frac{d^l F_{(i)}(x)}{dx^l} \right|_{x=a}, i \in [r]. \]

**Remark and example.** All we said before can be summed up in the following rule: The derivative \( f_{n_1, n_2, \ldots} := \frac{\partial^{n_1+n_2+\cdots} F(x_1, x_2, \ldots)}{\partial x_1^{n_1} \partial x_2^{n_2} \cdots} \) of a certain function \( F = F(x_1, x_2, \ldots) \) in the point \((x_1, x_2, \ldots)\) is the coefficient of \( t_1^{n_1} t_2^{n_2} \cdots \) in the expansion of \( f = f(t_1, t_2, \ldots) := F(x_1 + t_1, x_2 + t_2, \ldots) \) by any known method.

For example, if \( F = (x_1 + x_2)^{n_1} (x_1 + x_3)^{n_2} (x_2 + x_3)^{n_3} \), where \( a_1, a_2, a_3 \) are real numbers, we find by abbreviating \( \xi := x_1 + x_3, \) \( \xi := x_2 + x_3, \)

\[ f = f(t_1, t_2, t_3) = (x_1 + t_2 + x_3 + t_3)^{a_1} \times (x_2 + t_1 + x_1 + t_1)^{a_2} \times (x_3 + t_1 + x_2 + t_2)^{a_3} \]

\[ = F_1 \left( 1 + \frac{t_2}{\xi_1} + \frac{t_3}{\xi_2} \right)^{a_1} \left( 1 + \frac{t_1}{\xi_3} + \frac{t_1}{\xi_2} \right)^{a_2} \left( 1 + \frac{t_1}{\xi_3} + \frac{t_2}{\xi_3} \right)^{a_3}, \]

that we can expand by [12m] (p. 41) (be aware of the multinomial notation, [10c], p. 271):

\[ f = \sum_{k_1 + k_2 + k_3 \geq 0} \binom{a_1}{k_1} \binom{a_2}{k_2} \binom{a_3}{k_3} \times \prod_{k_1, k_2, k_3 \geq 0} \frac{\partial^{k_1+k_2+k_3} F_1^{k_1+k_2+k_3}}{k_1! k_2! k_3!} \]

Finally, taking the coefficient of \( r_1^n r_2^{r_2} \cdots / n_1! n_2! \cdots \), we obtain:

\[ f_{n_1, n_2} = \frac{\partial^{n_1+n_2} F_2}{\partial x_1^{n_1} \partial x_2^{n_2} \partial x_3^{n_3} \partial x_4^{n_4}} \sum_{k_1 \geq 0} \binom{n_1}{k_1} \binom{n_2}{k_2} \binom{n_3}{k_3} \times \prod_{k_4 \geq 0} \frac{\partial^{k_4} F_1}{k_4!} \left( x_1 + t_{k_1-k_2-k_3-k_4} \right) \]

3.3. Bell polynomials

**Definition.** The (exponential) partial Bell polynomials are the polynomials \( B_{n,k} = B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \) in an infinite number of variables \( x_1, x_2, \ldots \), defined by the formal double series expansion:

\[ \Phi = \Phi(t, u) := \exp \left( u \sum_{m > 0} x_m \frac{t^m}{m!} \right) = \sum_{n \geq 0} \frac{u^n}{n!} B_{n,k} t^k = \]

\[ = 1 + \sum_{n \geq 1} \frac{u^n}{n!} \left[ \sum_{k=1}^n u^k \right] B_{n,k} \{ x_1, x_2, \ldots \} \]

or, what amounts to the same, by the series expansion:

\[ \sum_{n \geq 1} \frac{u^n}{n!} B_{n,k} t^k \]
The (exponential) complete Bell polynomials $Y_n=Y_n(x_1, x_2, \ldots, x_n)$ are defined by:

$$\Phi(t, 1) = \exp\left(\sum_{m \geq 1} x_m \frac{t^m}{m!}\right) = 1 + \sum_{n \geq 1} Y_n(x_1, x_2, \ldots) \frac{t^n}{n!},$$

in other words:

$$Y_n = \sum_{k=1}^{n} B_{n,k}, \quad Y_0 := 1.$$
The following relations can be proved easily ($n \geq 1$):

\[ [3k] \quad k \mathbf{B}_{n,k} = \sum_{l=k+1}^{n-1} \binom{n}{l} x_{n-l} \mathbf{B}_{l,k-l} \]

\[ [3l] \quad \mathbf{B}_{n,k}(x_1, x_2, \ldots) = \sum_{l=k+1}^{n} \binom{n}{l} x_1^l \mathbf{B}_{l,k-l-1}(0, x_2, x_3, \ldots) \]

\[ = \sum_{l=0}^{k} \frac{n!}{l!(n-k)!} x_1^l \mathbf{B}_{n-k,k-l}(x_2, x_3, \frac{2}{3}, \ldots) \]

\[ [3l'] \quad \mathbf{B}_{n,k}(x, x_3, \frac{2}{3}, \ldots) = \frac{n!}{(k+n)!} \mathbf{B}_{n+k,k}(0, x_2, x_3, \ldots) \]

\[ [3l''] \quad \mathbf{B}_{n,k}(x_{a+1}, x_{a+2}, \frac{a+1}{a+2}, \frac{a+2}{a+3}, \ldots) = \frac{n!}{(n+qk)!} \mathbf{B}_{n+q,k}(0, x_2, x_3, \ldots) \]

\[ [3m] \quad B_{n,n-a}(x_1, x_2, \ldots) = \sum_{j=a+1}^{2a} \binom{n}{j} x_1^j B_{j,n-a}(0, x_2, x_3, \ldots) \]

\[ = \sum_{j=a+1}^{2a} \frac{n!}{(n-j)! a!} x_1^j \mathbf{B}_{n-j,n-a}(x_2, x_3, \frac{2}{3}, \ldots) \]

\[ [3n] \quad B_{n,k}(x_1 + x', x_2 + x_2', \ldots) = \sum_{x \in k, y \in n} \binom{n}{y} B_{n,k}(x_1, x_2, \ldots) \mathbf{B}_{n-k,n}(x', x_2', \ldots) \]

\[ [3n'] \quad B_{n,k}(0, 0, \ldots, 0, x_j, 0, \ldots) = 0, \quad \text{except} \quad B_{j,k} = \frac{(j)!}{k!(j)!} x_j. \]

Remark. The $B_{n,k}$, as given by [3a, a'], will give a simple way of writing the Taylor coefficients (= successive derivatives) of the formal series that we now are going to study. Meanwhile, if one works with ordinary coefficients, as on pp. 36-43, it is better to use the polynomials $\mathbf{B}_{n,k}$ (still with integral coefficients), defined by [30, a'] instead of [3a, a'] (and tabulated on p. 309):

\[ [3o] \quad \hat{F}(t, u) := \exp\left(\sum_{m \geq 1} x_m u^m\right) = \sum_{k \geq 1} \mathbf{B}_{n,k}(x_1, x_2, \ldots) t^m \frac{u^k}{k!} \]

\[ [3o'] \quad \left(\sum_{m \geq k} x_m u^m\right)^k = \sum_{m \geq k} \mathbf{B}_{n,k} t^m \]

that we call ordinary, in contrast to the $B_{n,k}$ already introduced, that we called exponential. More generally, just as in the case of the GF, [13a] (p. 44), let $\Omega_1, \Omega_2, \ldots$ be a reference sequence, $\Omega_1 = 1, \Omega_k \neq 0$, given once and for all; the Bell polynomials with respect to $\Omega$, $B_{n,k}^\Omega - \sum_{m \geq k} B_{n,k}^\Omega (x_1, x_2, \ldots)$ are defined as follows:

\[ [3p'] \quad \Omega_m (\sum_{m \geq 1} \Omega_m x_m t^m)^k = \sum_{m \geq k} B_{n,k}^\Omega t^m \]

($\Omega_{m+1}/m$ in the 'exponential' case, and $\Omega_m - 1$ in the 'ordinary' case).

\[ B_{1,1}^Q = x_1; \quad B_{2,1}^Q = x_2, \quad B_{2,2}^Q = x_1^2; \quad B_{3,1}^Q = x_3; \quad B_{3,2}^Q = 2\Omega_2^2 x_1 x_2, \quad B_{3,3}^Q = = x_3^3; \ldots \]

Meanwhile, it should be perfectly clear, once and for all, that the polynomials $B_{n,k}$ which occur in the sequel of this book always mean the exponential Bell polynomials ([3d] p. 134), unless explicitly stated otherwise.

### 3.4. Substitution of One Formal Series into Another; Formula of Faa di Bruno

**Theorem A (Faa di Bruno formula).** ([Faa di Bruno, 1855, 1857]. See also [*Bertrand, 1864 I, p. 138, [Cesaro, 1885], [Dederick, 1926], [Francais, 1815], [Marchand, 1886], [Teixeira, 1880], [Wall, 1938].]

Let $f$ and $g$ be two formal (Taylor) series:

\[ [fa] \quad f := \sum_{k \geq 0} \frac{f_k}{k!} u^k, \quad g := \sum_{m \geq 0} g_m r^m, \quad \text{with} \quad g_0 = 0, \]

and let $h$ be the formal (Taylor) series of the composition of $g$ by $f$ (Theorem C, p. 40):

\[ [fa] \quad h := \sum_{n \geq 0} \frac{h_n}{n!} r^n = f \circ g = f[g]. \]

Hence, the coefficients $h_n$ are given by the following expression:

\[ [fa] \quad h_n = 0, \quad h_n = \sum_{1 \leq j \leq n} f'_j B_{n,j-1}(g_1, g_2, \ldots, g_{n+j}), \]

where the $B_{n,k}$ are the exponential Bell polynomials ([3d] p. 134).

By definition [4b] of $h$, it is clear that the $\hat{h}_n$ are linear combinations of the $f_j$:

\[ [fa] \quad \hat{h}_n = \sum_{1 \leq j \leq n} A_{n,k} f_k, \]
and that the $A_{n,k}$ only depend on $g_1, g_2, \ldots$. Now these $A_{n,k}$ are determined by choosing for $f(u)$ the special formal series $f^*(u) := \exp(a u)$, where $a$ is a new indeterminate. Then:

$$[4e] \quad f^*_k = \left. \frac{d^k f^*}{du^k} \right|_{u=0} = a^k.$$ 

Hence, by [3a] (p. 133), for (1), and by [4d] for (2):

$$[4f] \quad h^* := f^* \circ g = \exp(a g) = \exp \left( \sum_{n \geq 1} g_n \frac{t^n}{n!} \right)$$

$$= 1 + \sum_{k \geq 1} B_{n,k} (g_1, g_2, \ldots) \frac{t^n}{n!} a^k.$$ 

$$[4g] \quad \sum_{n \geq 0} h_n^* \frac{t^n}{n!} = 1 + \sum_{n \geq 1} \left\{ \frac{t^{n}}{n!} \sum_{k=1}^{n} A_{n,k} j_k^* \right\}$$

from which it follows that $A_{n,k} = B_{n,k}$ by identifying the last members of $[4f]$ and $[4g]$. 

So, we find (see p. 307): $h_1 = f_1 g_1$, $h_2 = f_1 g_2 + f_2 g_1$, $h_3 = f_1 g_3 + f_3 g_1$, $h_n = f_1 g_n + f_n g_1 + f_2 g_2 + f_3 g_3 + \cdots + f_n g_n$.

By the Faà di Bruno formula we can effectively calculate the successive derivatives of a function of a function.

**THEOREM B.** Let two functions $F(y)$ and $G(x)$ of a real variable be given, $G(x)$ of class $C^\infty$ in $x=a$, and $F(y)$ of class $C^\infty$ in $y=b=G(a)$, and let $H(x) := (F \circ G)(x) = F(G(a))$. If we put:

$$[4h] \quad g_m := \left. \frac{d^m G}{dx^m} \right|_{x=a}, \quad f_k := \left. \frac{d^k f}{dy^k} \right|_{y=b}, \quad h_n := \left. \frac{d^n H}{dx^n} \right|_{x=a};$$

$$g_0 := G(a), \quad f_0 := F(b) = h_0 := H(a) = F[G(a)],$$

and we define the associated formal Taylor series:

$$g(t) := \sum_{m \geq 1} g_m t^m / (m!), \quad f(u) := \sum_{k \geq 0} f_k u^k / (k!),$$

$$h(t) := \sum_{n \geq 0} h_n t^n / (n!),$$

then we have formally: $h = f \circ g$. (Be careful! For $g$, the summation begins at $m=1$, so there is no constant term.)

If the Taylor expansions are convergent for $a$ and $t$ real, $|t| < R$, then we have: $H(a+t) = h(t) = F(b + g(t)) = \sum_{k \geq 0} f_k g^k(t) / (k!) = (f \circ g)(t)$. If there is no convergence, then operate with expansions of $f$ and $g$ considered as asymptotic expansions.

**THEOREM C.** Notations and hypotheses as in Theorem B for the functions $F$, $G$, $H = F \circ G$. Then the $n$-th order derivative of $H$ in $x = a$, $n \geq 1$, equals:

$$[4i] \quad h_n := \left. \frac{d^n h}{dx^n} \right|_{x=a} = \sum_{k=1}^{n} f_k B_{n,k} (g_1, g_2, \ldots, g_{n-k+1}),$$

where the $B_{n,k}$ are given explicitly by [3d].

Apply Theorems A and B.  

**Example.** What is the $n$th derivative of $F(x) = x^{a}\exp？(x > 0$ and $a$ is any fixed real number $\neq 0$). We can make the same observation as on p. 133. So, we must expand $f(t) := F(x + t)$ as a power series in $t$. Now, after a few manipulations:

$$f(t) = (x + t)^{a}\exp = F(x) \cdot \exp(a t \log x) \cdot \exp \left( \frac{x t}{x} \right) \log \left( 1 + \frac{t}{x} \right).$$

Let us introduce the integers $b(n, k)$ such that

$$\frac{1}{k!} ((1 + T) \log(1 + T))^k := \sum_{n \geq k} b(n, k) \frac{T^n}{n!}, \quad b(0, 0) := 1.$$ 

It is easy to verify: $b(n, k) = nb(n-1, k-1) + b(n, k-1) + (k-n) \times b(n, k)$, hence the following tableau for $b(n, k)$:

$$\begin{array}{ccccccccccc}
1/2/3/4/5/6/7/8/9/10
\end{array}$$

$$\begin{array}{ccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}$$
Moreover, \( b(n, k) = \sum_{l} \binom{1}{k} k^{l-k} s(n, l) \) with the Stirling numbers \( s(n, l) \) of p. 50.

Returning to \( f(t) \), we get consequently:

\[
f(n) = F(x) \sum_{j \geq 0, \xi \leq n} \frac{(at \log x)^j}{j!} b(m, k) \frac{t^m}{m!} (ax)^k.
\]

Finally, collecting the coefficients of \( t^n/n! \) in \( f(t) \) and abbreviating \( \lambda := \log x, \xi := (ax)^{-1} \), we obtain the following formula for the \( n \)-th derivative:

\[
f_n = a^n (x^{\lambda}) = a^n x^{\lambda} \sum_{j=0}^{n} \binom{n}{j} \lambda^j \sum_{h=0}^{n} b(n-j, n-h-j) \xi^h.
\]

For instance, \( f_4 = a^4 x^{\lambda} \{1 + 6 \xi - \xi^2 + 2 \xi^3 + 4 \lambda (1 + 3 \xi - \xi^2) + 6 \lambda^2 (1 + \xi) + + 4 \lambda^3 + \lambda^4\} \).

3.5. **Logarithmic and Potential Polynomials**

The following are three examples of applications of the Faà di Bruno formula.

**Theorem A** (successive derivatives of log \( G \)). The logarithmic polynomials \( L_n \) defined by:

\[
[5a] \quad \log \left( \sum_{n \geq 0} g_n n! \right) = \log \left( 1 + g_1 t + g_2 \frac{t^2}{2!} + \cdots \right) = \sum_{n \geq 1} L_n \frac{t^n}{n!} \quad (g_0 = 1),
\]

which are expressions for the \( n \)-th derivative of \( \log[G(x)] \) in the point \( x=a \), equal (for the notation, cf. \[3d\] p. 134 and \[4h\] p. 138):

\[
[5b] \quad L_n = L_n (g_1, g_2, \ldots, g_n) = \sum_{1 \leq k \leq \xi \leq n} (-1)^{k-1} (k-1)! B_{n-k} (g_1, g_2, \ldots). \quad (L_0 = 0)
\]

- Use \[4c, i\] with \( F(y) = \log y, \ b = 1, \ f_k = (-1)^{k-1} (k-1)! \).

From \[5a, b\] the following expansion is easily deduced:

\[
[5c] \quad \log \left( g_0 + g_1 t + g_2 \frac{t^2}{2!} + \cdots \right) = \log g_0 + \sum_{n \geq 1} L_n \frac{t^n}{n!} \left\{ \sum_{1 \leq k \leq n} (-1)^{k-1} (k-1)! g_0^{-k} B_{n-k} (g_1, g_2, \ldots) \right\}.
\]

where \( g_0 > 0 \). A table of logarithmic polynomials is given on p. 308. (On this subject, see also [Bouwkamp, De Bruijn, 1969].)

**Theorem B** (successive derivatives of \( g \)). The potential polynomials \( p_n(r) \) defined for each complex number \( r \) by:

\[
[5d] \quad \left( \sum_{n \geq 0} g_n n! \right)^r = \left( 1 + g_1 t + g_2 \frac{t^2}{2!} + \cdots \right)^r = 1 + \sum_{n \geq 1} p_n(r) \frac{t^n}{n!} \quad (g_0 = 1),
\]

which are expressions for the \( n \)-th derivative of \( G(x) \) in the point \( x=a \), equal (notations as in \[3d\] p. 134, and \[4h\] p. 138):

\[
[5e] \quad p_n(r) = \sum_{1 \leq k \leq \xi \leq n} (r)_k B_{n-k} (g_1, g_2, \ldots).
\]

- Use \[4c, i\] with \( F(y) = y, \ b = 1, \ f_k = (r)_k \).

From \[5d, e\] we obtain easily the expansion:

\[
[5f] \quad \left( g_0 + g_1 t + g_2 \frac{t^2}{2!} + \cdots \right)^r = g_0^r + \sum_{n \geq 1} p_n(r) \frac{t^n}{n!} \left\{ \sum_{1 \leq k \leq n} \left( r \right)_k g_0^{-k} B_{n-k} (g_1, g_2, \ldots) \right\}.
\]

where \( g_0 > 0 \) for \( r \) an arbitrary real or complex number, \( g_0 \neq 0 \) for \( r \) an arbitrary integer, and \( g_0 \) arbitrary for \( r \) an integer \( > 0 \). When \( g_0=0 \) in \[5f\], and \( r \) is an integer \( > 0 \), then we find back \[3a'\] (p. 133), and when \( r \) is integer \( < 0 \), we get the following Laurent series, whose expansion is given by \[5d\] \( (a, \neq 0) \):

\[
[5g] \quad \left( g_1 t + g_2 \frac{t^2}{2!} + \cdots \right)^r = (g_1)^r \left( 1 + \frac{g_2}{2 g_1} t \frac{1}{1!} + \frac{g_3}{3 g_1} \frac{t^2}{2!} + \cdots \right).
\]
Finally, by \([31^*]\), one may show that for all integers \(l\) and \(q \geq 0\), we have
\[
\left( g_q \frac{t^l}{q!} + g_{q+1} \frac{t^{l+1}}{(q+1)!} + \cdots \right)^{-1} = \\
= \frac{(g_1)^l}{(g_1)^l \cdot \sum_{m \geq 0} \sum_{j \in \mathbb{N}} \frac{(-1)^j (g_1)^j (g_q)^j}{(m + qj)! (g_q)^j} D_{m+qj, j} (0, 0, \ldots, 0, g_{q+1}, g_{q+2}, \ldots)}. 
\]

**Theorem C.** For any complex number \(r\), we have:
\[ P_a^{-n} = r \sum_{1 \leq j \leq n} (-1)^j \left( \frac{n}{r+j} \right) P_a^{(j)}. \]

In other words, for \(G(x) \in \mathbb{C}^\infty\) in the point \(a\), \(g_0 = G(a) = 1\):
\[ [S_j] \quad g_{n+1} = r \sum_{1 \leq j \leq n} (-1)^j \left( \frac{n}{r+j} \right) \frac{d^n}{dx^n} G'(x) \bigg|_{x=a}. \]

Let \(g = 1 + \sum_{n \geq 1} g_n x^n/(n!)\); then we get
\[ [S_j] \quad g^{r} = 1 + \sum_{n \geq 1} P_n^{(r-1/n!)} = n \sum_{k>0} \left( \frac{r}{k} \right) (g-1)^k. \]

Now \(r^k\) divides \((g-1)^k\); hence, by virtue of \([S_j]\), \(P_n^{(-r)}\) equals the coefficient of \(r^n/(n!)\) in:
\[ \sum_{k \geq 0} \binom{n}{k} (g-1)^k = \sum_{0 \leq j \leq k \leq n} \binom{n}{k} \binom{k}{j} (-1)^{n-j} g^j. \]

Hence
\[ P_n^{(-r)} = \sum_{0 \leq j \leq k \leq n} \binom{-r}{k} \binom{k}{j} (-1)^{n-j} P_n^{(j)} = \\
= \sum_{0 \leq j \leq n} (-1)^j P_n^{(j)} r^j. \]

where, using \([7g]\) (p. 17), for (*):
\[ y = \sum_{j \leq k \leq n} \binom{r + k - 1}{j} \binom{k}{j} = \left( \frac{r}{j} \right) \sum_{j \leq k \leq n} \binom{r + k - 1}{k-j} = \\
= \sum_{j \leq k \leq n} \binom{r + j - 1}{j} \binom{r + n}{n-j} = \frac{r}{j} \binom{n + r}{n-j} \binom{n}{j}. \]

3.6. Inversion Formulas and Matrix Calculus

We just treat two examples and for the rest we refer to [*Riordan, 1968*, pp. 43-127, for a very extensive study of the subject.

(1) Binomial Coefficients

Let two sequences be given, consisting, for instance of real numbers (more generally, in a commutative ring with identity) such that:
\[ [6a] \quad f_n = \sum_{0 \leq k \leq n} \binom{n}{k} g_k, \quad n \geq 0. \]

We want to express \(g_a \) as a function of the \(f_a \).

The simplest method consists of observing that \([6a]\) means that:
\[ [6b] \quad F = PG, \]
where \(F, G\) are matrices consisting of a single (infinite) column, and \(P\) the (infinite triangular) Pascal matrix:
\[ [6c] \quad F := \begin{pmatrix} f_0 \\ f_1 \\ \vdots \end{pmatrix}, \quad G := \begin{pmatrix} g_0 \\ g_1 \\ \vdots \end{pmatrix}, \quad P := \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 0 & 1 & 2 & 1 & \cdots \\ 0 & 0 & 1 & 3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}. \]

We take for \(F\) and \(G\) special matrices such that \(f_n + y^n, g_n - x^n\); in this case we get, by \([6a]\), \(y = 1 + x\). Hence \(x^n = (y-1)^n = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} y^k; \) consequently:
\[ [6d] \quad P^{-1} = \begin{pmatrix} (-1)^{n-k} \binom{n}{k} \end{pmatrix}_{k \geq 0} = \begin{pmatrix} 1 & -1 & 1 & 1 & \cdots \\ -1 & -2 & 1 & 3 & 3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}. \]
So, $P^{-1}$ is the same as $P$, except that signs $-$ appear in a chessboard pattern. (Because $P$ is triangular, [6d] also holds, if the matrices are cut off at the $n$-th line, and thus turned into finite matrices.) Finally, if we take into account that $G = P^{-1}F$

\[ 6e \quad g_n = \sum_{0 \leq k \leq n} (-1)^{n-k} \binom{n}{k} f_k. \]

(II) Stirling numbers

We now show that the matrix $s := [s(n, k)]_{n, k \geq 0}$ consisting of the Stirling numbers of the first kind, is the inverse of the matrix $S := [S(n, k)]_{n, k \geq 0}$ of the Stirling numbers of the second kind; this means, like in the preceding case of the binomial coefficients:

\[ 6f \quad f_n = \sum_k S(n, k) g_k \iff g_n = \sum_k s(n, k) f_k. \]

Now, using [14s] (p. 51) for (*), and using the notation:

\[ f := \sum_{m \geq 0} f_m t^m / m!, \quad g := \sum_{n \geq 0} g_n t^n / n!, \]

we get:

\[ 6g \quad f = f(t) = \sum_{m \geq 0} \frac{t^m}{m!} \left( \sum_k S(m, k) g_k \right) = \sum_{k \geq 0} g_k \left( \sum_{m \geq k} \frac{t^m}{m!} \binom{m}{k} \right) = \sum_{k \geq 0} g_k \frac{(e^t - 1)^k}{k!} = g(e^t - 1). \]

Putting $u := e^t - 1$, let $t = \log(1 + u)$. Then [6g] gives, with [14r] (p. 51) for (**):

\[ 6h \quad g = g(u) = f(\log(1 + u)) = \sum_{k \geq 0} f_k \frac{\log^k(1 + u)}{k!} = \sum_{k > 0} f_k \left( \sum_{n \geq k} \frac{s(n, k) u^n}{n!} \right) = \sum_{n \geq 0} \frac{u^n}{n!} \left( \sum_k s(n, k) f_k \right), \]

which proves [6f], if we identify the coefficients of $u^n / n!$ of the first and the last member of [6h].

3.7. Fractionary iterates of formal series

The Faà di Bruno formula, [4c] (p. 137), with $f = g$, gives the coefficients or derivatives of $f \circ f$, and more generally, it also gives the coefficients of

the iterate of order $\alpha$ of the formal series $f$ (when $f_0 = 0$, $\alpha$ integer $\geq 1$), denoted by $f^{(\alpha)}$, and defined as follows:

\[ 7a \quad f^{(1)} = f, \quad f^{(2)} = f \circ f, \ldots, f^{(\alpha)} = f \circ f^{(\alpha - 1)}. \]

We now want to define the iterate (analytical or fractionary) of order $\alpha$ of $f$, also denoted by $f^{(\alpha)}$, for any $\alpha$ from the field of the coefficients of $f$; in the case we consider, this will be the field of the complex numbers (this constitutes no serious loss of generality). In this section every formal series $f$ is supposed to be of the form:

\[ 7b \quad f = \sum_{n \geq 1} \Omega_n f^n, \]

where $\Omega_1, \Omega_2, \ldots$ is a reference sequence, given once and for all, $\Omega_1 = 1$, $\Omega_2 \neq 0$ (p. 44); in this way we treat at the same time the case of 'ordinary' coefficients of $f$ (resp. $\Omega_n = 1$), and the case of 'Taylor coefficients' (resp. $\Omega_n = 1 / n!$).

With every series $f$ we associate the infinite lower iteration matrix (with respect to $\Omega$):

\[ 7c \quad f = \sum_{n \geq 0} B_n \Omega_n e^n, \]

where $B_n = B_n^\Omega (f_1, f_2, \ldots)$ is the Bell polynomial with respect to $\Omega$ ([3p'] p. 137), defined as follows:

\[ 7d \quad \Omega_0 e^r = \sum_{n \geq k} B_n \Omega_n e^n. \]

Thus, the matrix of the binomial coefficients is the iteration matrix for

\[ f = t(1 - t)^{-1}, \quad \Omega_n = 1, \quad f = e^t - 1, \quad \Omega_n = 1 / n! \].

Theorem A. For three sequences $f, g, h$ (written as in [7b]) $h = f \circ g$ is equivalent to the matrix equality:

\[ 7e \quad B(h) = B(g) \cdot B(f). \]

([Jabotinski, 1947, 1949, 1963]. If we transpose the matrices, we get $h = f \circ g \iff B(h) = B(f \cdot B(g))$, which looks better. However, the classical
combinatorial matrices, as the binomial and the Stirling matrices, are most frequently denoted as lower triangular matrices, hence our choice. 

For each integer \( k \geq 1 \), we have, with \([7d]\) for (\(*\)):

\[
\sum_{\eta \geq 1} B_{\eta, k} (h_1, h_2, \ldots) \Omega_\eta r^\eta = \sum_{\eta \geq 1} B_{\eta, k} (f_1, f_2, \ldots) \Omega_\eta g^\eta =
\]

from which \([7c]\) follows if we collect the coefficient of \( \Omega_\eta r^\eta \) at both 'ends' of \([7f]\).

If we consider in \([7e]\) the first column of \( B(h) \) only, we obtain again the formula of Faà di Bruno ([4i] p. 139), if we take \( \Omega_\eta = 1/n! \). More generally, if we have \( n \) series \( f_1, f_2, \ldots, f_\eta \), then \([7e]\) gives the matrix equality \( B(f_\eta) = B(f_1) B(f_2) \ldots B(f_\eta) \). In other words, if we consider again the first column only, we obtain a generalized Faà di Bruno formula for the \( n \)-th derivative of the composite of \( \eta \) functions (again, we must take \( \Omega_\eta = 1/n! \)). Similarly, \( B(f^{(\alpha)}) = (B(f))^\alpha \) for all integers \( \alpha \geq 1 \), which leads to an explicit formula for integral order iterates ([Tambs, 1927])

Now we suppose that the coefficient of \( t \) in \( f \) equals 1, \( f_1 = 1 \); shortwise, we say that \( f \) is unitary. Furthermore, we assign values to \( B^\alpha \) as \( (B(f))^\alpha \), a complex, in the following way: denoting the unit matrix by \( I \), and putting \( B^\alpha = B-I \) (which is \( B \) with all 1's on the diagonal erased), we define:

\[
[B^\alpha]_{n, k} = \sum_{\eta \geq 1} \left( \begin{array}{c} \alpha \\ \eta \end{array} \right) [\mathcal{A}]_{n, \eta}^\alpha,
\]

by which the matrix \( B^\alpha \) can actually be computed. For all \( \alpha, \alpha' \), the reader will verify the matrix equalities:

\[
[B^\alpha B^{\alpha'}]_{n} = B^{\alpha + \alpha'} = B^{\alpha'} B^\alpha, \quad (B^\alpha)^\eta = B^{\eta \alpha} = (B^{\alpha})^\eta.
\]

For each complex number \( a \), the \( \alpha \)-th order fractional iterate \( f^{(\alpha)} \) of the unitary series \( f \) is the unitary series, whose iteration matrix is \( B^\alpha \). In other words, \( f^{(\alpha)} = \sum_{n \geq 1} f_n^{(\alpha)} \Omega_\eta r^\eta \), where the coefficients \( f_n^{(\alpha)} \) have the following expression, using \( b_{n, j} = [\mathcal{B}]_{n, 1}, n \geq 2; f_1^{(\alpha)} = 1 \).

Series \( f^{(\alpha)} \), thus defined, does not depend on the reference sequence \( \Omega_\eta \).

Evidently, \( f^{(0)} \) is the 'identity' series, \( f^{(0)}(t) = t \). In the case of 'Taylor coefficients', \( \Omega_\eta = 1/n! \), we obtain, by computing the powers \( \mathcal{B} \), the following first values for the iteration polynomials \( b_{n, j} \):

\[
\begin{align*}
b_{2, 1} &= 2f_2 + 3f_1^2, \\
b_{3, 2} &= 3f_3 + 5f_2f_1, \\
b_{4, 3} &= 4f_4 + 5f_3f_1 + 6f_2f_1^2, \\
b_{5, 4} &= 5f_5 + 5f_4f_1 + 5f_3f_1^2 + 6f_2f_1^3, \\
b_{6, 5} &= 6f_6 + 5f_5f_1 + 5f_4f_1^2 + 5f_3f_1^3 + 6f_2f_1^4, \\
b_{7, 6} &= 7f_7 + 6f_6f_1 + 5f_5f_1^2 + 5f_4f_1^3 + 5f_3f_1^4 + 6f_2f_1^5, \\
b_{8, 7} &= 8f_8 + 7f_7f_1 + 6f_6f_1^2 + 5f_5f_1^3 + 5f_4f_1^4 + 5f_3f_1^5 + 6f_2f_1^6.
\end{align*}
\]

From these values we obtain immediately, by \([7j]\), the expressions for the first derivatives \( f_n^{(\alpha)} \) of the iterate \( f^{(\alpha)} \). For example, the fractional iterate of \( f(t) = e^t - 1 = \sum_{n \geq 1} t^n/n! \) is \( f^{(\alpha)}(t) = t + \sum_{n \geq 2} f_n^{(\alpha)} t^n/n! \), where \( f_n^{(\alpha)} = \sum_{j=1}^{\alpha} \left( \begin{array}{c} \alpha \\ j \end{array} \right) b_{n, j} \) for \( n \geq 2 \); the first few values of \( b_{n, j} \) are:

\[
\begin{align*}
b_{2, 1} &= 2f_2 + 3f_1^2, \\
b_{3, 2} &= 3f_3 + 5f_2f_1, \\
b_{4, 3} &= 4f_4 + 5f_3f_1 + 6f_2f_1^2, \\
b_{5, 4} &= 5f_5 + 5f_4f_1 + 5f_3f_1^2 + 6f_2f_1^3, \\
b_{6, 5} &= 6f_6 + 5f_5f_1 + 5f_4f_1^2 + 5f_3f_1^3 + 6f_2f_1^4, \\
b_{7, 6} &= 7f_7 + 6f_6f_1 + 5f_5f_1^2 + 5f_4f_1^3 + 5f_3f_1^4 + 6f_2f_1^5, \\
b_{8, 7} &= 8f_8 + 7f_7f_1 + 6f_6f_1^2 + 5f_5f_1^3 + 5f_4f_1^4 + 5f_3f_1^5 + 6f_2f_1^6.
\end{align*}
\]
Evidently, the alternating row sums \( \sum_{i=1}^{n-1} (-1)^i b_{n,i} \) equal \((-1)^{n-1} \times (n-1)!\), since \( f_n^{-1} (t) = \log(1+t) \).

**Theorem B.** For all complex numbers \( z, z' \), the fractional iterates of the unitary series \( f \) satisfy:

\[
\begin{align*}
\left[7k\right] & \quad f^{(z+z')} = f^{(z)} \circ f^{(z')}, \\
& \quad (f^{(z')})^{(z)} - f^{(z+z')} = (f^{(z)})^{(z')}. \\
\end{align*}
\]

This follows immediately from [7i].

---

### 3.8. Inversion formula of Lagrange

For every formal series \( f = \sum_{n=0}^{\infty} a_n t^n \), we denote the derivative by \( f' \) or \( Df \), or \( df/dt \); let furthermore:

\[
\left[8a\right] \quad C_n f := a_n = \text{the coefficient of } t^n \text{ in } f.
\]

Supposing \( a_0 = 0, a_1 \neq 0 \), we are going to compute the coefficients \( a_n^{(-1)} \) of the reciprocal series, which is:

\[
f^{(-1)} = \sum_{n \geq 1} a_n^{(-1)} t^n,
\]

such that \( f \circ f^{(-1)} = f^{(-1)} \circ f = t \) (inversion problem for formal series).

**Theorem A.** (inversion formula of Lagrange). With the notation \[8a\], we have, for all integers \( k, 1 \leq k \leq n \):

\[
\left[8b\right] \quad C_n (f^{(-1)})^k = \sum_{k \leq n} \left( \frac{t^k}{n!} \right) t^{n-k}.
\]

([Lagrange, 1770]. See also [Lagrange, Legendre (Bürmann), 1799]. The formal demonstration given here is due to [Henrici, 1964]. There is an immense literature on this problem, and we mention only [Blakley, 1964a, b, c], [Brun, 1955], [Good, 1960, 1965], [*Gröbner, 1960] p. 50-68, [Perceus, 1964], [Raney, 1960, 1964], [Sack, 1963a, b, 1966], [Stieltjes, 1885], [Tyrrell, 1962].) In (8b), \( (f/t)^{-n} \) means evidently \( a^{-n} (1 + (a_2/a_1) t + (a_3/a_1) t^2 + \ldots) \).

According to Theorem A (p. 145), all we need to prove is that the product of the matrix whose \( n \)-th row-\( k \)-th column coefficient is the right-hand member of \[8b\], by the matrix whose \( n \)-th row-\( k \)-th column coefficient is \( C_n f^k \) (this is the matrix \( B(f) \), with respect to \( O_{n+1} \), [7c], p. 145), equals the identity matrix \( I \). Now, the coefficient on the \( n \)-th row and \( k \)-th column, say \( \pi_{n,k} \), of this product matrix, is by definition equal to:

\[
\pi_{n,k} := \sum_{k \leq n} \left( \frac{t^k}{n!} \right) t^{n-k}.
\]

So we only have to prove that \( \pi_{n,k} = 1 \) for \( n = k \) and \( = 0 \) for \( n \neq k \). For this, we observe that \( t C_n f^k = C_n (t D(f^k)) = k C_n (t f^{k-1} f') \). Hence, with \[12g\] (p. 37) for (*):

\[
\pi_{n,k} = \sum_{k \leq n} \left( \frac{t^k}{n!} \right) t^{n-k} = \left( \frac{t}{n!} \right) t^{n-k} f^{(k-1)} \left( f \right) = \frac{k}{n} C_n \left( t f^{k-1} f' \right).
\]

which implies immediately that \( \pi_{n,k} = 1 \), for \( n = 1, 2, \ldots \). For \( n > k \), on the other hand, we have:

\[
\pi_{n,k} = \frac{k}{n} C_n \left( t D \left( \frac{f^{n+k}}{n+k} \right) \right),
\]

where the series following the differentiation sign \( D \) is now a Laurent series (p. 43). In the derivative of such a series terms \( t^{-1} \) cannot occur, so indeed \( \pi_{n,k} = 0 \).

Here are other forms of the Lagrange formula \[8b\].

**Theorem B.** With notations as above, and \( u := f^{(-1)} (t) \) we have for any
formal series $\Phi$:

$$\Phi(u) = \Phi(0) + \sum_{n \geq 1} \frac{u^n}{n!} \frac{\partial^n}{\partial t^n} \Phi'(t) \left( \frac{f(t)}{t} \right)^{-n}$$

or, if one likes that more:

$$n \Phi_{n} \Phi(f^{(-1)}(t)) = \frac{\partial^n}{\partial t^n} \Phi'(t) \left( \frac{f(t)}{t} \right)^{-n}.$$  

Let $\Phi(v) := \sum_{k \geq 0} \varphi_k v^k$; it suffices to show [8c] for $\varphi_k^k$; but this is just [8b].

**Theorem C.** Let $y = y_0 + x F(y)$ determine $y$ as a series in $x$, with constant term $y_0$. Then:

$$\Xi(y) = \Xi(y_0) + \sum_{n \geq 1} \frac{x^n}{n!} \frac{\partial^n}{\partial y^n} \Xi(y_0) F^n(y_0).$$

Writing $y = y_0 + u$, we get $x = u(F(y_0 + u))^{-1} = f(u)$. Then apply [8c], with $t = x, \Phi(u) = \Xi(y_0 + u)$.

**Theorem D.** ([Hermite, 1891]). With notations as above, and $u = f^{(-1)}(t)$, we have for all formal series $\Psi$:

$$\frac{t \Psi'(u)}{u f'(u)} = \sum_{n \geq 0} \frac{1}{n!} \frac{\partial^n}{\partial t^n} \Psi(t) \left( \frac{f(t)}{t} \right)^{-n}.$$  

in other words:

$$C_n \frac{t \Psi'(u)}{u f'(u)} = C_n \Psi(t) \left( \frac{f(t)}{t} \right)^{-n}.$$  

If we take the derivative of [8c] with respect to $t$, then, using $t = f(u), du/dt = 1/f'(u)$, we get:

$$\Phi'(u) \frac{du}{dt} = \Phi'(u) \sum_{n \geq 0} \frac{1}{n!} \frac{\partial^n}{\partial t^n} \Phi'(t) \left( \frac{f(t)}{t} \right)^{-n-1}.$$  

So we only need to substitute $\Psi(u) := u \Phi'(u)/f(u)$ into [8e].

**Theorem E.** The Taylor coefficients of the formal series $f^{(-1)} = \sum_{n \geq 1} f^{(-1)} u^n/n!$, which is the reciprocal of $f = \sum_{n \geq 1} f_n u^n/n!$ can be expressed as function of the Taylor coefficients $f_n$ of $f$ in the following manner:

$$f_n^{(-1)} = \sum_{k=1}^{n-1} (-n)_k f_{n-k} B_{n-k, 1} \left( f_2, f_3, \ldots \right)$$

$$= \sum_{k=1}^{n-1} (-1)^k f_{n-k} B_{k+n-1, k}(0, f_2, f_3, \ldots)$$

with $f_2 = 1/f_1$, and with $B_{n,k}$ the exponential Bell polynomials. ([3d], p. 134. For this problem see also [Böedewadt, 1942], [Kamber, 1946], [Ostrowski, 1957] and [*1966], p. 235, [*Riordan, 1968], pp. 148 and 177.)

[8f] is an immediate consequence of [8b], with $k = 1$, where the right-hand member is expressed by means of [5f] (p. 141); then [8g] follows from [3f] (p. 136).

The first values of $f_n^{(-1)}$ are:

$$f_1^{(-1)} = - f_1 + 1 \quad f_2^{(-1)} = - f_2 f_1 + 1 = - 1 \quad f_3^{(-1)} = - f_3 f_2 f_1 + 1 = - 1 \quad f_4^{(-1)} = - f_4 f_3 f_2 f_1 + 1 = - 1 \quad f_5^{(-1)} = - f_5 f_4 f_3 f_2 f_1 + 1 = - 1 \quad f_6^{(-1)} = - f_6 f_5 f_4 f_3 f_2 f_1 + 1 = - 1 \quad f_7^{(-1)} = - f_7 f_6 f_5 f_4 f_3 f_2 f_1 + 1 = - 1 \quad f_8^{(-1)} = - f_8 f_7 f_6 f_5 f_4 f_3 f_2 f_1 + 1 = - 1 \quad f_9^{(-1)} = - f_9 f_8 f_7 f_6 f_5 f_4 f_3 f_2 f_1 + 1 = - 1 \quad f_{10}^{(-1)} = - f_{10} f_9 f_8 f_7 f_6 f_5 f_4 f_3 f_2 f_1 + 1 = - 1 \quad f_{11}^{(-1)} = - f_{11} f_{10} f_9 f_8 f_7 f_6 f_5 f_4 f_3 f_2 f_1 + 1 = - 1 \quad f_{12}^{(-1)} = - f_{12} f_{11} f_{10} f_9 f_8 f_7 f_6 f_5 f_4 f_3 f_2 f_1 + 1 = - 1 \quad f_{13}^{(-1)} = - f_{13} f_{12} f_{11} f_{10} f_9 f_8 f_7 f_6 f_5 f_4 f_3 f_2 f_1 + 1 = - 1 \quad f_{14}^{(-1)} = - f_{14} f_{13} f_{12} f_{11} f_{10} f_9 f_8 f_7 f_6 f_5 f_4 f_3 f_2 f_1 + 1 = - 1 \quad f_{15}^{(-1)} = - f_{15} f_{14} f_{13} f_{12} f_{11} f_{10} f_9 f_8 f_7 f_6 f_5 f_4 f_3 f_2 f_1 + 1 = - 1 \quad f_{16}^{(-1)} = - f_{16} f_{15} f_{14} f_{13} f_{12} f_{11} f_{10} f_9 f_8 f_7 f_6 f_5 f_4 f_3 f_2 f_1 + 1 = - 1 \quad f_{17}^{(-1)} = - f_{17} f_{16} f_{15} f_{14} f_{13} f_{12} f_{11} f_{10} f_9 f_8 f_7 f_6 f_5 f_4 f_3 f_2 f_1 + 1 = - 1 \quad f_{18}^{(-1)} = - f_{18} f_{17} f_{16} f_{15} f_{14} f_{13} f_{12} f_{11} f_{10} f_9 f_8 f_7 f_6 f_5 f_4 f_3 f_2 f_1 + 1 = - 1 \quad f_{19}^{(-1)} = - f_{19} f_{18} f_{17} f_{16} f_{15} f_{14} f_{13} f_{12} f_{11} f_{10} f_9 f_8 f_7 f_6 f_5 f_4 f_3 f_2 f_1 + 1 = - 1 \quad f_{20}^{(-1)} = - f_{20} f_{19} f_{18} f_{17} f_{16} f_{15} f_{14} f_{13} f_{12} f_{11} f_{10} f_9 f_8 f_7 f_6 f_5 f_4 f_3 f_2 f_1 + 1 = - 1 \quad f_{21}^{(-1)} = - f_{21} f_{20} f_{19} f_{18} f_{17} f_{16} f_{15} f_{14} f_{13} f_{12} f_{11} f_{10} f_9 f_8 f_7 f_6 f_5 f_4 f_3 f_2 f_1 + 1 = - 1 \quad f_{22}^{(-1)} = - f_{22} f_{21} f_{20} f_{19} f_{18} f_{17} f_{16} f_{15} f_{14} f_{13} f_{12} f_{11} f_{10} f_9 f_8 f_7 f_6 f_5 f_4 f_3 f_2 f_1 + 1 = - 1$$

To check this table, observe that the coefficient of $(-1)^k f_{n-k}$, when $f_1 = f_2 = \cdots = 1$, is exactly $S_k(k+n-1, k)$ of p. 222.

**Theorem F.** Let $a$ be an integer $\geq 1$. For $f(t) = t(1 - \sum_{m \geq 1} x_m t^{am}/m!)$, we have $f^{(-1)}(t) = t(1 + \sum_{m \geq 1} y_m t^{am}/m!)$, where

$$y_m = \sum_{k=1}^{m} (am + k)_{k-1} B_{m,k}(x_1, x_2, \ldots).$$

Apply [8b] (p. 148).
up to $t^{13}$, we need the $B_{n,k}$ up to $n=12$ by [8f], and only up to $n=3$ by [8h]. So, $f^{-1}(t)=t-t^{5}/30+t^{9}/22680-t^{13}/97297200+\ldots$ ([Zyczkowski, 1965]).

**Theorem G.** We have the following formula, using only coefficients of powers of $f(t)$ with positive integral exponents ($f(t)=a_1 t+a_2 t^2+\cdots$, $a_1 \neq 0$):

$$C_m(f^{-1}(t))^k = \frac{k}{n} \sum_{j=1}^{n-k} \binom{n}{n-j} \binom{n-k}{j} \times a_{n-k+j} C_{n-j}, \quad (f(t))^j.$$  

Use [8b] (p. 148) and [5h] (p. 142).

**Remark.** The correspondence between a formal series and its iteration matrix was already used when we inverted the Stirling matrix $S$ (p. 144): we took the inverse function of $f(t):=e^t-1$, whose iteration matrix was $S$ (with respect to $\Omega_m=1/n!$)

**Applications**

(I) The most classical example is undoubtedly that of computing the coefficients of the inverse function $f^{-1}(t)$ for the case $f(t)=te^{-t}$. By [8b] (p. 148), $k=1$, we get:

$$C_m f^{-1} = \frac{1}{n} C_m \binom{te^{-t}}{t} = \frac{1}{n} C_m \epsilon^n = \frac{1}{n} \frac{n^{n-1}}{n!}.$$  

Hence $f^{-1}(t)=\sum_{m=1}^{\infty} \frac{n^{n-1} t^m}{m!}$. (See also Exercise 18, p. 163.)

(II) For given fixed complex $z$, what is the 'value' of the series $F(t):=\sum_{n=0}^{\infty} \binom{nt}{n} t^n$? Since $F(t)=\sum_{n=0}^{\infty} t^n C_n (1+t)^n$.

we can apply [8d] with $f(t)=(1+t)^n$ and $\Psi(t)=1$. After simplifications, we obtain $F(t)=(1+u) \left(1-(1-u) t^{-1}\right)$. where $u:=f^{-1}(t)$ is the reciprocal of $f(t)$. (For $z=2$ we find back (1) of Exercise 22, p. 81.)

(III) Calculate the $n$-th derivative of an implicit function. We consider a Taylor formal expansion in two variables: $f(x,y)=\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f_{m,n} x^m y^n (m!n!)$. where $f_{0,0}=0, f_{0,1}\neq 0$. Therefore, $f(x,y)=\sum_{m=1}^{\infty} \phi_a(x) y^n/m!$, with $\phi_a(x)=\sum_{n=0}^{\infty} f_{m,n} x^n/m!$. We want to find a formal series $y(1+x)^{-n}$ such that $f(x,y)=0$ (the problem of 'implicit functions'). For that, we solve $\sum_{n\geq 1} \phi_a(x)^n/n!=x_0$ by the Lagrange formula, where the variable is $-x_0$, the unknown function is $y$, all the $\phi_1, \phi_2, \phi_3, \ldots$ being temporarily considered as constants, and collect afterwards the terms in $x^n/n!$ in the expression of $y$ just found, where $\phi_0=\phi_0(x), \phi_1=\phi_1(x), \ldots$. Putting $a:=f_0, b=-(f_0)^{-1}$, we find ([Comtet, 1968], [David, 1887], [Goursat, 1904], [Sack, 1966]. [Teixeira, 1904], [Warontzoff, 1894]) and p. 175): $y_1=ab$ (this is the well-known formula $y'_n=f^{-1}_{n+1} f^{-1}_{n+2}$).

(IV) Solve the equation $y-x = x^a y^b$, where $p$ and $q$ are integers $\geq 0$. We have $x=y(1-x^p y^q)=f(y)$. So, with [8b] p. 148, $y=\sum_{n=0}^{\infty} b_n x^n$, where $b_n=b_n(x)=(1/n) C_{n+1} (1-x^{p+1})^{-n}$.

Therefore,

$$y = x \sum_{k=0}^{\infty} \frac{q(k+1)}{k} (kq+k)x^{(p+q)}.$$

(V) Let us give another proof of Abel formula ([11b] p. 128). For that, take $f(t)=te^{t}, \Phi(t)=e^{xt}$ in [8c]. Then $\Phi(u)=e^{xt}=1+\sum_{k=1}^{\infty} \frac{t^k}{k!} x^k$. Now, multiply the preceding by $e^{xt}$, replace $x$ by $f(u)=ue^{xt}$, and take coefficient of $u^n/n!$.

### 3.9. Finite summation formulas

Now we want, in the simplest cases, to express a sum $A:=\sum_{k=0}^{\infty} a(k)$ by means of an explicit (or closed) formula, called a summation formula, that is an expression in which the summation sign $\sum$ does not occur anymore (neither little dots!).

**Example 1.** Show that $A:=\sum_{k=0}^{\infty} \binom{n}{k} = 2^n$. In fact, $A=(1+1)^n$, because of the binomial formula.

**Example 2.** Compute $A_n(x):=\sum_{k=0}^{\infty} \binom{n}{k} x^k$. We have $\sum_k \binom{n}{k} x^k = (1+x)^n$. Taking the derivative, we get $\sum_k k \binom{n}{k} x^{k-1} = n(1+x)^{n-1}$. 


Hence \( A_n(x) = n x (1 + x)^{n-1} \). Particularly, \( A_n(1) = \sum k \binom{n}{k} = n 2^{n-1} \) and \( A_n(-1) = \sum (-1)^k k \binom{n}{k} = 0 \), except \( A_1(-1) = 1 \).

**Example 3.** Compute \( A := \sum \binom{n}{k} k \). Observe that \( A = \sum_{k=0}^{n} \binom{n}{k} \times \binom{n}{n-k} \), which means that \( A \) equals the coefficient of \( t^n \) in the product of \((1 + t)^n \) with itself:

\[
A = \binom{n}{n} (1 + t)^n (1 + t)^n = \binom{n}{n} (1 + t)^{2n}.
\]

(See Exercise 38, p. 90.) More generally, we have the convolution identity of Vandermonde:

\[
[9a] \sum_{k=0}^{n} \binom{n}{k} \binom{n-m}{k-h} = \binom{n}{m}, \quad k, m \leq n.
\]

which follows from p. 26 or [13c] on p. 44, or also, as before, from:

\[
\binom{n}{k} = \binom{n-1}{k} (1 + t)^n = \binom{n-1}{k} (1 + t)^n (1 + t)^{-m}.
\]

In other cases, \( A = A(n) = \sum_{k=1}^{n} a(k) \) and a summation formula expresses now that \( A = \sum b(l) \), where \( b(l) \) is another sequence. If \( m < n \), we save making additions in this way. More generally, a summation formula is an equality between two expressions, one of which contains one or more summations. A summation formula is interesting if it establishes a connection between expressions which are built up from known or tabulated expressions.

**Example 4.** Use the Bernoulli polynomials ([14a], p. 48), to compute for each integer \( r \geq 0 \):

\[
[9b] Z - Z(n, r) := \sum_{1 \leq i \leq n} k^r = 1^r + 2^r + \cdots + n^r.
\]

For this we consider the formal series:

\[
f_a(t) := \sum_{r=0}^{\infty} \{Z(n, r) \frac{t^r}{r!}\}.
\]

We get, by [14a] (p. 48), for (1):

\[
[9c] f_a(t) = t \sum_{r=0}^{\infty} \binom{r}{0} \frac{k^r}{r!} = t \sum_{1 \leq i \leq n} \left( \sum_{r=0}^{\infty} \frac{(kt)^r}{r!} \right) = t \sum_{1 \leq i \leq n} e^{kt}.
\]

Hence, by identification of the coefficient of \( t^{r+1}/r! \) in the first and last member of [9c], we get, by [14g] (p. 48), for (2), \( r \geq 1 \) (\( Z(n, 0) - n \)):

\[
[9d] Z(n, r) = \frac{1}{r+1} \left\{ B_{r+1}(n+1) - B_{r+1} \right\} = \frac{1}{r+1} \sum_{0 \leq j \leq r} B_j \frac{(r+1)!}{k!} (n+1)^{r+1-k}.
\]

Thus we find, by the table on p. 49 (a table of the \( Z(n, r), r \leq 10, n \leq 100 \) is found in [*Abramovitz, Stegun, 1964*, pp. 813–17; see also [Carlitz, Riordan, 1963], Exercise 4, p. 220 and Exercise 31, p. 169]):

\[
Z(n, 1) = n(n + 1)/2,
Z(n, 2) = n(n + 1)(2n + 1)/6,
Z(n, 3) = n^2(n + 1)^2/4,
Z(n, 4) = n(n + 1)(2n + 1)(3n^2 + 3n - 1)/30,
Z(n, 5) = n^2(n + 1)^2(2n^2 + 2n - 1)/12,
Z(n, 6) = n(n + 1)(2n + 1)(3n^4 + 6n^3 - 3n + 1)/42,
Z(n, 7) = n^2(n + 1)^2(3n^4 + 6n^3 - 3n - 2)/24,
Z(n, 8) = n(n + 1)(2n + 1)(5n^6 + 15n^5 + 5n^4 - 15n^3 - n^2 + 9n - 3)/90.
\]

As additional properties of \( Z(n, r) \), we have:

\[
(1) \quad Z(n, r) = r^{n} Z(n, r-1) + B_{r+1} \]

\[
(2) \quad Z(n, 2) \text{ divides } Z(n, 2k) \text{ and } Z(n, 3) \text{ divides } Z(n, 2k+1),
\]

\[ k \geq 1. \]

**SUPPLEMENT AND EXERCISES**

1. **Two relatives of the binomial identity.** Show that:

\[
(x + y)^{2n} = \sum_{k=0}^{n} \binom{2n}{n-k} (x^k + y^k) (x + y)^{n-k}.
\]
\[ x^n + y^n = \sum_{0 \leq k \leq n/2} (-1)^k \binom{n-k}{k} (x+y)^n \times k^n. \]

[Hint: Induction. See also Exercise 35, p. 87 and p. 198.]

4. Lah numbers ([*Riordan, 1958], p. 43). These are the numbers
\[ L_{n,k} = (-1)^n \binom{n}{k} \frac{n!}{k!} \] which appeared in [3h] (p. 135), exp \{tu\times 
(1-t)^{-1}\} = 1 + \sum_{k \geq 1} L_{n,k} (-t)^n n!/n!.
(1) \[ L_{n+1,k} = -(n+k) L_{n,k} \]

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-\[ L_{n,k-1}. \]

2. \((x)_n = (-1)^n (x)_n = \sum_{k=0}^n \binom{n}{k} L_{n,k} (x)_k.\]

3. Bell, potential and logarithmic polynomials. (1) Show that \( k! B_{n,k} = \sum_{r=1}^n \binom{n}{r} P_n^{(r)}. \) Which property of derivatives does this formula give when combined with the Faà di Bruno formula of p. 137? (2) Use \( \log (1+g) = \sum_{r \geq 1} (-1)^{r-1} g^r \), where \( g = \sum_{r \geq 1} a_r x^r/n! \) to show that \( B_n = \sum_{r=1}^n (-1)^{r-1} g^r R_n^{(r)}. \) Translate this formula in terms of derivatives. Similarly, with \( s(l,k) \), the Stirling number of the first kind:

\[ \log^k (1+g) = \sum_{l=k}^n \frac{s(l,k)}{k!} (\log x)^l. \]

4. \( P_n^{(r)} \) as a function of a single Bell polynomial when \( r \) is integer. If \( r \)

is a positive integer, show that:

\[ P_n^{(r)} = \binom{n+r}{r}^{-1} B_{n+r,r}(1, 2y_1, 3y_2, \ldots). \]

[Hint: We get \( (1 + g_1 + g_2 t^2/2! + \cdots)^r = t^{-r} (1 + 2g_1 t^2/2! + 3g_2 t^3/3! + \cdots)^r \) by [5g], p. 141.]

5. Determinantal expressions. (1) Let \( f = \sum_{a \geq 0} a_n x^n, a_0 \neq 0, \) and \( g = -\sum_{a \geq 0} a^a x^n = -f^{-1}. \) Then \( b_n = (-1)^n a_n^{a_n-1} \det [c_{i,j}], \) where \( c_{i,j} = : a_j, 1 \leq i, j \leq n; a_j = 0 \) for \( k < 0. \) (This gives a determinantal expression for \( P_n^{(r-1)}. \) (2) The Faà di Bruno formula ([4h] p. 139) can be restated operationally in the following form ([Ivanoff, 1958]), using the Pascal triangle of dimension \( n, \) with an upper diagonal of \(-1: \)

\[ h_n = \begin{cases} g_1 D & -1 \quad 0 \quad 0 \quad \ldots \quad 0 \\ g_2 D & g_1 D & -1 \quad 0 \quad \ldots \\ g_3 D & 2g_2 D & g_1 D & -1 \quad \ldots \\ & \vdots & \vdots & \vdots & \ddots & \end{cases} f, \]

where \( D^r f = f_k. \) For example,

\[ h_3 = \begin{pmatrix} g_1 D & -1 \quad g_1 D \\ g_2 D & g_1 D \\ g_3 D & 2g_2 D & g_1 D \\ \vdots & \vdots & \vdots & \ddots & \end{pmatrix} f = (a_1 D^2 + a_2 D) f = a_1 f_2 + a_2 f_1. \]

6. Successive derivatives of \( F(\log x) \) and \( F(e^x). \) Expressed as a function of the Stirling numbers of the first kind \( s(n,k) \) and of the second kind \( S(n,k) \) we have:

\[ \frac{d^n}{dx^n} F(\log x) = x^{-n} \sum_{k=0}^n s(n,k) F^{(k)}(\log x) \]

\[ \frac{d^n}{dx^n} F(e^x) = \sum_{k=1}^n S(n,k) e^{kx} F^{(k)}(e^x) \]

Moreover, for \( y = x_1 x_2 \ldots x_n, \) we have

\[ \frac{\partial^n F(y)}{\partial x_1 \partial x_2 \ldots \partial x_n} = \sum_{k=1}^n S(n,k) y^{k-1} F^{(k)}(y). \]

7. Successive derivatives of \( F(x^\alpha). \) Let \( \alpha \) be a real constant and \( F(x) \) a function of class \( C^\infty \) in the point \( x = a (>0). \) Using the notations of [4h]
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(p. 138), and the Faà di Bruno formula [4i] of p. 139, show that the $n$-th derivative of $H(x):=F(x^a)$ in the point $x=a$ equals $h_n=\sum_{k=1}^n f(a)\frac{d^a}{dx^a}Z_{n,k}(x)$, where the $Z_{n,k}(x)$ are generated by $((1+T)^{x}-1)^{k}=(\prod_{i=1}^k Z_{n,k}(a))^{T/n!}$ (See Exercise 21, p. 163.)

Deduce the well-known formulas:

$$Z_{n,k}(-1)=(-1)^{n} \frac{n!}{k!(n-k)!}$$
$$Z_{n,k}(1)=(-1)^{n-k} \frac{(n-1)!}{(k-1)!} \frac{(2n-k-1)}{2^{2n-k}}$$
$$Z_{n,k}(2)=\frac{n!}{k!} \left( \frac{k}{n-k} \right) 2^{2k-n}.$$

8. Expansions of the coordinates with respect to the Frenet-Serret trihedron in terms of arclength. Let $\varrho=\varrho(s)$ be the curvature of a plane curve $M=M(s)$ as a function of the length $s$ of the arc with origin $M(0)$ (intrinsic equation).

We introduce the Frenet-Serret trihedron $(M(0), t, n, \varrho)$, where $\varrho=\frac{dM}{ds}|_{s=0}$, $\varrho>0$, and $M(0)(s)=x(s)+y(s)\varrho=\sum_{k=1}^n x_k s^{n-k}/n!$, $y(s)=\sum_{k=1}^n y_k s^{n-k}/n!$. Putting $\varrho_k=\frac{d\varrho}{ds}|_{s=0}$, $\varrho(0)=\varrho(s)$, $B_{n,k}=B_{n,k}(\varrho_0, \varrho_1, \varrho_2, \ldots)$, we have:

$$x_{n+1} = \sum_{k=1}^n (B_{n,k}-B_{n,k+1}),$$
$$y_{n+1} = \sum_{k=1}^n (B_{n,k}-B_{n,k+1}).$$

For example, $x_1=1, x_2=0, x_3=-\varrho_0, x_4=-3\varrho_0\varrho_1, x_5=-4\varrho_0\varrho_2-3\varrho_2^2, \ldots, y_1=0, y_2=\varrho_0, y_3=\varrho_0, y_4=\varrho_2-\varrho_0^2/2, y_5=\varrho_3-6\varrho_0\varrho_2+\varrho_2^2, \ldots$

* Find similar formulas for a space curve with respect to the curvature $\varrho=\varrho(s)$ and the torsion $\tau=\tau(s)$.

9. Symmetric functions. A symmetric function, abbreviated SF, is a polynomial $P(x_1, x_2, \ldots, x_n)$ in the $n$ variables $x_1, x_2, \ldots, x_n$, with coefficients in a field $K$ (often $K=R$ or $K=C$), and which is invariant under any permutation of the variables: for any $\sigma \in \mathfrak{S}(n)$, $P(x_1, x_2, \ldots, x_n)=P(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. A monomial symmetric function (abbreviated MSF) is a symmetric function of the form:

$$f=\sum_{i} x_i^{q_1} x_i^{q_2} \ldots x_i^{q_n}$$
also denoted by $\sum^{\ell}(x_i^{q_1} x_i^{q_2} \ldots x_i^{q_n})$.

where the $q_i$ are given integers such that $q_i \geq q_2 \geq \cdots \geq q_n \geq 1$, and where the above summation takes place over all $\ell$-arrangements $(i_1, i_2, \ldots, i_\ell)$ of $[n]$ such that the corresponding monomials (in the summation) are all distinct. Thus $\sum^{\ell}(x_i^{q_1} x_i^{q_2} \ldots x_i^{q_n})$ are the 'elementary SF' and the 'sum of $r$-th powers SF', respectively. (1) Every SF is a linear combination of MSF (detailed tables in [*David, Kendall, Barton, 1966]).

Particularly $(x_1+x_2+x_3+\cdots+x_n)^r$ is a linear combination of MSF; in this summation occur $p(w)$ such MSF, which is the number of partitions of $w$ (pp. 94 and 126). (2) The $s_i$ have for GF: $P(t)=\sum_{j=0}^{\infty} s_j t^j=\prod_{r=1}^{\infty} (1+t^r)(1+t^r+t^2r)$. (3) $s_i=(-1)^{n-i}/(r-1)!\log(1+t^r)$. (4) $s_i=((-1)^{n-i}/(r-1)!)\log(1+t^r)$. (5) $s_i=((-1)^{n-i}/(r-1)!)\log(1+t^r)$.

Finally, give formulas and recurrences for the D'Arcais numbers $A(n, k)$ defined by $((1-t)(1-t^2)(1-t^3)\cdots)^{-1}=\sum_{n=0}^{\infty} t^n/n!$ (D'Arcais, 1913), of which the first values are:

<table>
<thead>
<tr>
<th>n \backslash k</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<td></td>
<td></td>
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<tr>
<td>2</td>
<td>1</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td></td>
<td>1</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>59</td>
<td>18</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>144</td>
<td>215</td>
<td>30</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>1440</td>
<td>2475</td>
<td>565</td>
<td>45</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>7560</td>
<td>28294</td>
<td>9345</td>
<td>1225</td>
<td>63</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>7560</td>
<td>293292</td>
<td>34016</td>
<td>147889</td>
<td>27720</td>
<td>2338</td>
<td>84</td>
<td>1</td>
</tr>
</tbody>
</table>
11. Characteristic numbers for a random variable. Let be given a probability space \((\Omega, A, \mathbb{P})\) and a real random variable \(X: \Omega \rightarrow \mathbb{R}\) (abbreviated RV) with distribution function \(F(x) := \mathbb{P}(X < x)\). Let \(\mu_n\) (or \(\mu'_n\)) be the central (or noncentral) moments of \(X\): 
\[
\mu_n := \mathbb{E}(X^n) = \int_{\Omega} x^n d\mathbb{P}(x), \quad \mu'_n := \mathbb{E}(X^n) = \mathbb{E}(X^n - \mu)^n, \quad \mu = \mathbb{E}(X) = \mathbb{E}(X) \quad \text{is the expectation of} \ X \quad \text{then} \ \mu_0 = 0.
\]
We define furthermore for \(X\) the variance \(\mu_2 = \mathbb{E}(X^2 - \mu)^2\) (also denoted by \(\var{X}\)) and the standard deviation \(\mu_2 = \mathbb{E}(X - \mu)^2\); the GF of the moments:
\[
\Psi(t) := 1 + \sum_{n=1}^\infty \mu_n t^n/n! = \mathbb{E}(e^{tX});
\]
the generating function of the central moments:
\[
\Psi^*(t) := 1 + \sum_{n=2}^\infty \mu'_n t^n/n! = \mathbb{E}(e^{t(X - \mu)}) = e^{-\mu} \Psi(t);
\]
and the GF of the cumulants \(\chi_n\):
\[
\gamma(t) := \log \Psi(t) = \sum_{n=1}^\infty \chi_n t^n/n!.
\]
If the RV is discrete (\(X(\Omega) \subseteq \mathbb{N}\)), \(p_k = \mathbb{P}(X = k)\), then we have the GF of the probabilities: \(g(u) := \sum_{k \geq 0} p_k u^k\); hence \(g(e^t) = \Psi(t)\), \(g(e^t) = \gamma(t)\).

(1) \(\mu_n = \sum_{k=0}^n \binom{n}{k} (-1)^k \mu'_n-k\); \(\mu'_n = \sum_{k=0}^n \binom{n}{k} \mu^k \mu'_{n-k}\), where \(0 \leq k \leq n\), \(\mu_0 = \mu_0 =: 1\). (2) \(\mu'_n = \mu_n = \mathbb{Y}(x_1, x_2, \ldots, x_n)\), \(\mu_n = \mathbb{Y}(0, x_2, x_3, \ldots, \mu_n) = \mathbb{N}(0, \mu_2, \mu_3, \ldots)\). (3) Let \(X_1, X_2, X_3, \ldots\) be independent Bernoulli RV's with the same distribution law, \(P(X_1 = 0) = q, P(X_1 = 1) = p, q \geq 0, p + q = 1\). Then \(E(X_1 + X_2 + \cdots + X_n)^l = \sum_k (n)_k p^k S(l, k)\). (4) Let \(X\) be a Poisson RV, \(P(X) = e^{-\lambda} \lambda^k / k!\); \(\lambda > 0\) is called the parameter of \(X\). Then \(\mu'_n = \sum_k S(n, k) \lambda^k; \mu' = \mu = 2, \mu = 3 = \lambda, \mu_3 = \lambda + 3 \lambda^2, \mu_5 = \lambda + 10 \lambda^2, \mu_6 = \lambda + 25 \lambda^2 + 15 \lambda^3 \ldots\).

12. Factorial moments of a RV. With the notations of Exercise 11, we define for each discrete RV, \(p_k = \mathbb{P}(X = k)\), the factorial moments: \(\mu_{(m)} := \sum_{k=0}^n \mu_k S(n, k) p_k = k(k-1)(k-m+1) \ldots, \ p = 6, m = 1, 2, 3, \ldots\). Show that \(\mu_{(m)} = \sum_k S(m, k) \mu_k, \mu'_n = \sum_k S(m, k) \mu'_{(k)}\), and that \(g(1+t) = \sum_{m=0}^n \mu_{(m)} t^m/m!\).

13. Random formal series. Let \(X_1, X_2, \ldots\) be Bernoulli random variables with the same distribution function, \(P(X_1 = 1) = p, P(X_1 = 0) = 1 - p, 0 < p < 1\). Let \(V_1, V_2, \ldots, W_1, W_2, \ldots\) be the RV defined by \(exp(X_1 t + 0^2 t^2 + \cdots, -1 = 1, V_1 t + 1^2 t^2 + \cdots)\). Show that the expectations \(E(V_1)\) and \(E(W_1)\) tend to infinity with \(n\).

*14. Distribution of a sum of uniformly distributed RV. Let \(X_1, X_2, \ldots, X_n\) be independent symmetrical RV with uniform distribution function. In other words, there exist \(\alpha, \beta, v = 1, 2, \ldots, n \) such that \(|X_i| \leq \alpha, \beta, \) and, for \(x \in [-\alpha, \beta]\), \(P(X_i < x) = (x + \alpha)/(2\beta)\). Determine the distribution function of \(S = X_1 + X_2 + \cdots + X_n\) in other words \(P(S < x)\) ([Ostrowski, 1952]).

15. A formula of Halphen ([Halphen, 1879]). Use [8b] (p. 148) or some other way, to show that:
\[
\frac{d^n}{dx^n} \left\{ x^n F \left( \frac{1}{x} \right) \right\} = \frac{(-1)^n}{x^{n+1}} F^{(n)} \left( \frac{1}{x} \right),
\]
where \(F^{(n)}(1/x)\) stands for the \(n\)-th derivative of \(F\) taken in the point \(1/x\). Thus \(d^n/(dx^n) (x^n = \log x) = (n-1)!/(x^n) \log x = n! (\log x + 1 + 1 + \cdots + 1/n)\). (1) \(d^n/(dx^n) (x^n = \log x) = (n-1)!/(x^n) \log x = n! (\log x + 1 + 1 + \cdots + 1/n)\). (2) \(d^n/(dx^n) (x^n = \log x) = (n-1)!/(x^n) \log x = n! (\log x + 1 + 1 + \cdots + 1/n)\). (3) \(d^n/(dx^n) (x^n = \log x) = (n-1)!/(x^n) \log x = n! (\log x + 1 + 1 + \cdots + 1/n)\). (4) \(d^n/(dx^n) (x^n = \log x) = (n-1)!/(x^n) \log x = n! (\log x + 1 + 1 + \cdots + 1/n)\).

*16. Lambert series and the Mőbius function. Let \(f(t) = \sum_{n=1}^\infty a_n t^n\) and \(g(t) = \sum_{n=1}^\infty a_n t^n (1 - t^n)^{1/4}\), which is called the Lambert GF of the sequence \(a_n\). (1) We have \(g(t) = \sum_{n=1}^\infty f(t^n)\). (2) Defining the Mőbius function \((-sequence) \mu(n) = \sum_{d|n} \mu(d)\). (3) \(\mu(1) = 1\); furthermore, for \(n = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}\), where the \(p_i\) are distinct prime factors of \(n\), we have \(\mu(n) = (-1)^k\) if all \(a_i\) equal 1 (such numbers \(n\) are called squarefree), and \(\mu(n) = 0\) in the other cases. It follows that \(\mu(n)\) is multiplicative, in the sense that when \(a\) and \(b\) are relatively prime, then \(\mu(ab) = \mu(a) \mu(b)\).
Show that \( t + t^2 + t^4 + t^8 + \cdots = \sum_{m \geq 0} \mu(2m+1) t^{2m+1} (1 - t^{2m+1})^{-1}. \) (4)

Let \( d(n) \) be the number of divisors of \( n \), in other words the number of solutions with integers \( x \) and \( y \geq 1 \) of the equation \( xy = n \). Then \( \sum_{n \geq 1} d(n) t^n = \sum_{n \geq 1} t^n (1 - t^n) = \sum_{n \geq 1} t^n (1 + t^n) (1 - t^n)^{-1}. \) (5) If \( \varphi(n) \) is the indicator function of Euler, \([6e] \) p. 193, then we have \( t(1 - t)^{-2} = \sum_{n \geq 1} \varphi(n) t^n (1 - t^n)^{-1}. \) Moreover, \( \sum_{n \geq 1} \varphi(n) (1 + t^n - (1 + t^n) (1 - t^{2m+1})^{-1}. \)

(A generalization of Lambert series is found in \([6f] \) p. \(1957\).) \([\text{Hint: The } c_n \text{ are integers if and only if the } b_m \text{ defined inductively by } g(x) = \sum_{n \geq 1} (1 - x^n)^m, \text{ are all integers. Consider then } \log g(x), \text{ and expand } k a_k = - \sum_{m \geq 1} b_m. \] Then apply the Möbius inversion formula (2) of Exercise 16.)

18. With the Lagrange formula. (1) Deduce from \( x = y \exp(-y) \) that \( \exp(xy) = 1 + \sum_{n \geq 1} a(\alpha + n)x^n/n! \) and \((1-y)^{-1} \exp(xy) = \sum_{n \geq 0} (n + x)^x/n! \). (2) Supposing \( f(t) = f + a_2 t^2 + a_3 t^3 + \cdots (a_1 = 1) \), prove that for every complex number \( \alpha \), with \( k \leq n \):

\[
\sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \exp(\alpha) = \frac{k + \alpha}{n + \alpha} C_{n-k} \left( \frac{f(t)}{t} \right)^{-n - \alpha}.
\]

19. Middle trinomial coefficients. These are \( a_n = C_{n-1}(1 + t + t^2)^n \) (p. 77):

\[
\begin{array}{ccccccccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
 a_n & 1 & 1 & 3 & 7 & 19 & 51 & 141 & 393 & 1107 & 3139 & 8953 & 25653
\end{array}
\]

(1) The integer \( a_n \) is the number of distributions of indistinguishable balls into \( n \) different boxes, each box containing at most 2 balls. (2) \( (n+1) a_{n+1} = (2n+1) a_n + 3 n a_{n-1}. \) (3) \( \sum_{n \geq 0} a_n t^n = (1 - 2 - 3 t^2)^{-1/2}. \)

(4) 1 Using the notation \([6f] \) p. 110) \( \sum_{n \geq 1} \alpha a_n = [3^{n+1}/4]. \) (5) For \( n \to \infty \), we have the asymptotic equivalent \( a_n \sim 3^n \sqrt{3/(4 n \pi)}. \) (6) For each prime number \( p \), then \( a_n \equiv 1 (\mod p) \) holds.

20. Hurwitz identity \([\text{Hurwitz, 1902}] \). Considering the set \( E \) of acyclic functions of \( [n+2] \) whose set of roots is \( \{n+1, n+2\} \), prove, by an argument similar to that of p. 129:

\[
(x + y) (x + y + z_1 + z_2 + \cdots + z_n)^{n-1} - 1 = \sum_{i=1}^n x (x + e_1 z_1 + \cdots + e_{n-1} z_n) \cdots (x + e_{n-1} x) + \cdots + e_{n-1} x^{n-1},
\]

where the summation is over all \( 2^n \) choices of \( e_1, \ldots, e_n \) independently taking the values 0 and 1, and \( e_i := 1 - e_i. \) Generalize for more than 2 roots.

21. Expansions related to \( 1 - (1 - at)^k \). (1) When \( k \) and \( l \) are given
integers \( \geq 1 \), express the Taylor coefficients of \( f:=(1+x)^{1/n}-1 \) in the point \( x=0 \) by an exact formula of rank \( (l-2) \). (as defined on p. 216.

Such a formula is apparently only useful if \( k \geq 1 \). [Hint: Putting \( y:=(1+x)^{1/n}-1 \), we have \( x=(y+1)^n-1 \) and \( f=y^k \); hence [8d] (p. 150) can be applied.] (2) For any real number \( n \),

\[
\left( 1 + \frac{\sqrt{1-4t}}{2} \right)^n = \left( 1 + \frac{-\sqrt{1-4t}}{2} \right)^n = 1 + u \sum_{n>1} \left( \frac{u+2n-1}{n-1} \right)^n.
\]

(3) Using Hermite's formula ([8d] p. 150), prove that for any \( x \):

\[
\delta(n, k) = \left( \frac{1}{\alpha^k} \right)^{x^{n-1}} \left( \frac{1}{n} \right)^{x^n}.
\]

22. Three special triangular matrices. (Obviously, the three following computations of infinite lower triangular matrices give the same result if the matrices are truncated at the \( n \)-th row and column, so that they become square \( n \times n \) matrices.) We let \( \mu(n, k) \) denote the coefficient on the \( n \)-th row and the \( k \)-th column of the matrix \( M \), and we let \( \mu^{(o)}(n, k) \) denote the corresponding coefficient in the matrix \( M^o \) (in the sense of [7g] p. 146). (1) Let \( \mu(n, k) := \binom{n+k}{n-k} \) for \( 0 \leq k \leq n \) and \( :=0 \) otherwise. (That is the coefficient of \( (-1)^{x+k} k! \) in the Laguerre polynomial \( L_n^{(o)}(x) \) of p. 50.) Then \( \mu^{(-1)}(n, k) = (-1)^{n-k} \binom{n+k}{n-k} \). [Hint: Straightforward verification, or the method of GF, p. 144.] (2) Let \( \mu(n, k) := \binom{n}{k} \binom{k}{n-k} \) for \( 1 \leq k \leq n \) and \( :=0 \) otherwise. Then \( \mu^{(o)}(n, k) = (-1)^{n-k} \binom{n}{k} \binom{k}{n-k} \).

[Hint: [8b], p. 148. See also Exercise 43, p. 91] (3) Let \( f(t) = \sum_{m=0}^n a_m t^m \). We put \( \mu(n, k) := a_{n-k} \) for \( 0 \leq k \leq n \) and \( :=0 \) otherwise. Then \( \mu^{(o)}(n, k) = b_{n-k} \) for \( 0 \leq k \leq n \) and \( :=0 \) otherwise, where the \( b_m \) are defined by \( f^{(o)}(t) = \sum_{m=0}^n b_m t^m \).

23. 'Inversion' of some polynomials. \( B_n(x), P_n(x) \) and \( H_n(x) \) denote the Bernoulli ([14a] p. 48), the Legendre ([14l] p. 50), and the Hermite ([14n] p. 50) polynomials, respectively. Show that:

\[
x^o = \sum_{k=1}^n \binom{n}{k} (n-k+1)^{-1} B_k(x)
\]

\[
x^o = n! 2^{-n} \sum_{0 \leq t \leq n/2} (2n-4k+1) \frac{k!}{2^{n-k}} \binom{n}{k} B_{n-2k}(x)
\]

\[
x^o = n! 2^{-n} \sum_{0 \leq t \leq n/2} \frac{k! (n-2k)!}{2^{n-k}} H_{n-2k}(x).
\]

It is somewhat more difficult to invert the Gegenbauer and Laguerre polynomials of p. 50. [Hint: Lagrange formula.]

24. Coverings of a finite set. A covering \( \mathcal{B} \) of \( N, |N|=n \), is an unordered system of blocks of \( N, \mathcal{B} \subseteq \mathcal{B}^*(\mathcal{B}(N)) \), whose union equals \( N \cup \bigcup_{B \in \mathcal{B}} B = N \). The number \( r_n \) of coverings of \( N \) equals \( \sum 1 \times \binom{n}{k} 2^{n-k-1} \), \( r_1 = 1, r_2 = 5, r_3 = 109, r_4 = 32297, r_5 = 2147321017. \) [Hint: \( |\mathcal{B}^*(\mathcal{B}(N))| = 2^{n-1} - 1 = \sum \binom{n}{k} r_k \), and [6a, c], p. 143.] Also compute the number \( r_{n,m} \) of coverings with \( m \) blocks, \( |\mathcal{B}| = m \), and the number \( r_{n,b} \) of coverings with \( b \)-blocks \((B \in \mathcal{B} = |B| = b)\). ([Comtet, 1960]. See also Exercise 40, p. 303.)

25. Regular chains ([Schröder, 1870]). Let \( a \) be an integer \( \geq 2 \), and \( N \) a finite set, \( |N|=n \). We 'chain' now \( a \) elements of \( N \) together in a \( a \)-block \( A_1 \subseteq N \). Let \( N_1 \) be the set, whose \((n-a+1)\) elements are the \((n-a)\) elements of \( N \setminus A_1 \) and the block \( A_1 \). Then we chain again \( a \) elements of \( N_1 \) together into a block \( A_2 \), from which we obtain a new set \( N_2 \), etc. We want now to compute the total number of such chains, called regular chains, not taking the order of the chaining into account. Show first that:

\[
c_n = \frac{1}{a! k_1 + k_2 + \cdots + k_a = n-1} \frac{n!}{k_1! k_2! \cdots k_a!} c_1 c_2 \cdots c_a,
\]

where \( c_0:=0, c_1 = 1, c_2 = c_3 = \cdots = c_{a-1} = 0, c_a = 1. \) [Hint: Consider the \( a \)-blocks in existence just before the last chaining operation, in the case they are of size \( k_1, k_2, \ldots, k_a \).] Obtain from this \( \mathcal{C} = \mathcal{C}(t):=\)
:= \sum_{n \geq 0} c_n t^n n! = t + C^2 t^2, and also obtain the value of \( c_n \) by applying the inversion formula of Lagrange.

26. The number of connected graphs ([Ridell, Uhlenbeck, 1953], [Gilbert, 1956b]). A connected graph over \( N, |N| = n \), is a graph such that any two of its points are connected by at least one path (Definition B, p. 62). Let \( \tau(n, k) \) be the total number of graphs with \( n \) nodes and \( k \) edges, and \( \gamma(n, k) \) the number of those among them that are connected. Clearly, \( \tau(n, k) = \binom{n}{k} \). The connected component \( C(y) \) of a vertex \( y \in N \) is the set of all \( z \in N \) 'connected' to \( y \) by at least one path. Now we choose \( x \in N \), and let \( M := N \setminus \{x\} \). Giving a graph on \( N \) equivalent to giving the trace \( V \) of \( C(x) \) on \( M (C(x) = \{x\} + V) \), and to giving, moreover, a graph on \( M \setminus V \); show that:

\[ \tau(n, k) = \sum_{\gamma(n, k) \in \mathcal{P} \setminus \{x\}} \binom{n - 1}{v} \gamma(v + 1, w) \tau(n - 1 - v, k - w). \]

Deduce from this:

\[ \sum_{n, k \geq 0} \gamma(n, k) \frac{\tau}{n!} u^k = \log \left( 1 + \sum_{m \geq 1} \frac{(1 + u)^m}{m!} \right). \]

More generally, let \( \tau_\beta(n, k) \) be the number of graphs with \( n \) vertices and \( k \) edges such that each connected component has the property \( \beta \), and let \( \gamma_\beta(n, k) \) be the number of those among them that, moreover, are connected. Then:

\[ \sum_{n, k \geq 0} \gamma_\beta(n, k) \frac{\tau}{n!} u^k = \log \left( 1 + \sum_{l \geq 1} z_\beta(m, l) u^l \right). \]

27. Generating functions and computation of integrals ([Comtet, 1967]).

1. Let \( J_m := \int_0^{\pi/2} (A^2 \cos^2 \phi + B^2 \sin^2 \phi)^{-m} d\phi \). Then \( \sum_{m \geq 1} J_m t^m = \int_0^{\pi/2} (A^2 \cos^2 \phi + B^2 \sin^2 \phi - t)^{-1} d\phi = (\pi t/2) (A^2 - t) (B^2 - t)^{-1/2}. \)

By expanding this last function into a power series, deduce that \( J_{m+1} = \pi (2^{m+1} A B m!)^{-1} \sum_{s=0}^m a_m^s A^{2s} B^{2m-2s} \), where the coefficients \( a_m^s \) satisfy the recurrence relation \( a_{m+2, s} = (2m+3) (a_{m+1, s-1} + a_{m+1, s}) - 4m+1 a_{m, s-1} \). The first few values of the \( a_m^s \) are:

<table>
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<th>2</th>
<th>3</th>
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<td>1</td>
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<tr>
<td>( a_m^1 )</td>
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<tr>
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<td>1825</td>
<td>2715</td>
<td>3785</td>
<td>4955</td>
</tr>
</tbody>
</table>

2. Compute

\[ \int_{-\infty}^{\infty} \left( x^2 + a^2 \right) (x^2 + b^2)^{-m} dx \]

and

\[ \int_{-\infty}^{\infty} \left( \sum_{j=1}^{\infty} (x^2 + a_j^2) \right)^{-m} dx \]

3. Compute \( A_n := \int_0^{\pi/2} (\log \sin^2 \phi \cos^2 \phi)^n d\phi \), where \( \alpha \) and \( \beta \) are \( \geq 0 \) ([Chaudhuri, 1967]). [Hint:

\[ \sum_{n \geq 0} A_n \frac{t^n}{n!} = \int_0^{\pi/2} \sin^{2n+1} \phi \cos^{2n} \phi \, d\phi = \frac{1}{2} \frac{\Gamma((1 + \alpha)/2) \Gamma((1 + \beta)/2)}{\Gamma((1 + \alpha + \beta)/2)}. \]

4. Compute \( I(p, q) = \int_0^\infty (\log x)^r (1 + x^2)^{-p} \, dx \), where \( p \) and \( q \) are positive integers. [Hint: \( \sum_{n \geq 0} A_n \frac{t^n}{n!} = \int_0^{\pi/2} \sin^{2n+1} \phi \cos^{2n} \phi \, d\phi = \frac{1}{2} \frac{\Gamma((1 + \alpha/2) \Gamma((1 + \beta/2))}{\Gamma((1 + \alpha + \beta)/2)}. \]

28. A multiple series. Let \( S \) be the convergent series of order \( k \) defined by \( \sum \{c_1 c_2 \cdots c_k (c_1 + c_2 + \cdots + c_k) \}^{-1} \), where the summation is taken over all systems of integers \( c_1, c_2, \ldots, c_k \) which are all \( > 1 \) and relatively prime. Then \( S = k! \) (AMM 73 (1966) 1025).

29. Expansion of \( (\arcsin t)^r \). Use the Cauchy formulas:

\[ \sin n x = u \sum_{n \geq 0} (-1)^n (u^2 - 1^2) (u^2 - 2^2) \cdots (u^2 - (2n - 1)^2) \]

\[ \cos n x = \sum_{n \geq 0} (-1)^n (u^2 - 1^2) (u^2 - 2^2) \cdots (u^2 - (2n - 1)^2) \]

\[ \begin{align*}
\text{(2)} \quad & \int \left( x^2 + a^2 \right) (x^2 + b^2)^{-m} d x \\
\text{(3)} \quad & \int_0^{\pi/2} (\log \sin^2 \phi \cos^2 \phi)^n d \phi, \quad \alpha \text{ and } \beta \geq 0 \quad ([\text{Chaudhuri}, 1967]). \\
\text{(4)} \quad & \int_0^\infty (\log x)^r (1 + x^2)^{-p} \, dx, \quad p \text{ and } q \text{ positive integers.} \\
\end{align*} \]
where \( x = \arcsin t \) has to be substituted ([Teixeira, 1896]). Use the same formulas to prove:

\[
\frac{\sin ux}{\cos x} = 2 \sum_{n \geq 0} (-1)^n (u^2 - 2^2) \cdots (u^2 - (2n)^2) \left( \frac{\sin^{2n+1} x}{(2n+1)!} \right)
\]

\[
\frac{\cos ux}{\cos x} = \sum_{n \geq 0} (-1)^n (u^2 - 2^2) \cdots (u^2 - (2n)^2) \left( \frac{\sin^{2n} x}{(2n)!} \right)
\]

30. Some summation formulas and interesting combinatorial identities.

\[
\sum_{k=0}^{n} \frac{k}{(k+1)!} - \frac{1}{(n+1)!} = 0
\]

\[
\sum_{k=0}^{n} \frac{(-1)^k \binom{n-k}{k}}{k} 2^{n-2k} = n + 1
\]

\[
\sum_{k=0}^{n} \frac{n^k}{k} = (2n-2) \binom{n-1}{n}
\]

\[
\sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} = \sum_{i=1}^{n} i^{-1}
\]

\[
\sum_{i=0}^{n} \min(m, n) = \frac{1}{2} n(N + 1)(3M - N + 1)
\]

\[
\sum_{i=0}^{n} \max(m, n) = \frac{1}{2} n(N^2 - 1) + \frac{1}{2} MN(M + 1)
\]

\[
\sum_{k=1}^{n} k \cdot k! = (n + 1)! - 1,
\]

and its generalization (of Gould):

\[
\sum_{k=0}^{n} \frac{x^k}{k} \binom{k}{k+1} \left( \frac{1}{x^k+1} \right)^p - x^p = \left( \frac{x}{n+1} \right)^p \left( \frac{(n+1)!}{x^{n+1}} \right)^p - 1
\]

\[
\sum_{k=0}^{n} \frac{\binom{n}{k} x^k}{k} - \sum_{i=0}^{n} \binom{n}{i} \left( \frac{2n-1}{n} \right)(x-1)^i
\]

31. Sum of the \( r \)-th powers of the terms of an arithmetic progression.

Let \( S_r = \sum_{k=1}^{n} (a + (k-1)b)^r \). By a method analogous to that used on p. 154, find the value of \( S_r \) as a function of the Bernoulli numbers. One can also establish the recurrence relation \( (a+nb)^{r+1} = a^{r+1} + \sum_{l=1}^{r+1} \binom{r+1}{l} b^l S_{r+1-l} \), where \( S_0 := n \). [Hint: Consider \( \sum_{k=1}^{n} (a+kb)^{r+1} \) and expand then \( (a+kb)^{r+1} = b^r + (a+(k-1)b)^{r+1} \) using the binomial identity.]

As examples, for \( t_1 := n^2 + 3^3 + 5^5 + \cdots + (2n-1)^{2n-1} \), we find:

\[
t_1 = n^2, \quad t_2 = \left( \frac{2n+1}{3} \right), \quad t_3 = n^2(2n^2 - 1)
\]

32. Four trigonometric summation formulas ([Hofmann, 1959]). For \( r \) integer \( \geq 1 \), we have:

\[
\sum_{k=1}^{n} \sin^2 kx = 2^{2r+1} \left\{ 2 \binom{2r}{r} + \sum_{k=0}^{r} (-1)^k \binom{2r}{r-k} \frac{\sin[k(2n+1)x]}{\sin kx} \right\}
\]

\[
\sum_{k=1}^{n} \sin^{2r+1} kx = 2^{2r} \sum_{k=0}^{r} (-1)^k \left\{ 2 \binom{2r+1}{r-k} \times \frac{\sin(2k+1)(n+1)x}{\sin(2k+1)x/2} \right\}.
\]
33. On the roots of \( ax = \tan x \). For computing the root \( x \) which lies between \( n\pi \) and \((n+1)\pi\), insert \( x = n\pi + \pi/2 - u, |u| < \pi/2 \), in \( ax = \tan x \). Then, \( t := (ax(n+\pi/2))^{-1} = (a + au \tan u)^{-1} = f(u) \), which can be (formally) inverted by the Lagrange formula: \( u = f^{-1}(t) \). Returning to \( x \), the following purely asymptotic expansion holds:

\[
x \approx (n + \frac{1}{2})\pi - \sum_{m \geq 0} (-1)^m \frac{t^{2m+1}}{(2m+1)!!} \left\{ \sum_{k=0}^{m} \frac{(-1)^{m-k} C(m, k) a^k}{2(2m+1)} \right\},
\]

where the \( C(m, k) \), closely related to arctangent numbers (p. 260), satisfy:

\[
C(m, k) = \frac{(2m - 1)2m(2m + 1)}{(2m - k)(2m - k + 1)} \cdot \{C(m - 1, k - 1) + C(m - 1, k)\}.
\]

Here is a table of the \( C(m, k) \):

<table>
<thead>
<tr>
<th>( m \backslash k )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>15</td>
<td>15</td>
<td>945</td>
<td>10395</td>
<td>135135</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>20</td>
<td>20</td>
<td>525</td>
<td>2550</td>
<td>134950</td>
<td>2538212</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>20</td>
<td>525</td>
<td>525</td>
<td>525</td>
<td>134970</td>
<td>2395364</td>
<td>23953645</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>1548</td>
<td>8232</td>
<td>8232</td>
<td>17640</td>
<td>13230</td>
<td>94518579</td>
<td>135135</td>
</tr>
<tr>
<td>4</td>
<td>525</td>
<td>1548</td>
<td>8232</td>
<td>8232</td>
<td>17640</td>
<td>13230</td>
<td>94518579</td>
<td>135135</td>
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<tr>
<td>5</td>
<td>525</td>
<td>1548</td>
<td>8232</td>
<td>8232</td>
<td>17640</td>
<td>13230</td>
<td>94518579</td>
<td>135135</td>
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<tr>
<td>6</td>
<td>134950</td>
<td>2395364</td>
<td>23953645</td>
<td>23953645</td>
<td>23953645</td>
<td>23953645</td>
<td>23953645</td>
<td>23953645</td>
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<td>135135</td>
<td>3963105</td>
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<td>981245265</td>
<td>1938871935</td>
<td>2029052025</td>
<td>869593725</td>
</tr>
</tbody>
</table>

Of course, when \( a = 1 \), \( x = \tan x \), the alternating horizontal sums extend Euler's result: \( x = (n + \frac{1}{2})\pi - \sum_{m \geq 0} c_m t^{2m+1}((2m+1)!!) \), where \( t = (\pi(n+\frac{1}{2}))^{-1} \) and

\[
c_m = (-1)^m \sum_{k=0}^{m} \frac{(-1)^{m-k} C(m, k)}{2(2m+1)} a^k.
\]

34. About the (purely) formal series \( \varphi(t) = \sum_{n \geq 1} n! t^n \). Let us define the integers \( A(n, k) \) by \( \varphi(t)^k = \sum_{n \geq k} A(n, k) t^n \). (1) These numbers satisfy the following recurrence: \( A(n, k) = A(n-1, k-1) + ((n+k-1)/k) x A(n-1, k) \). (2) Also find a triangular recurrence for the \( A(n, k) \) verify the following tables (of course, \( u = A(n, k) \))

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>-2</td>
<td>1</td>
<td>-4</td>
<td>1</td>
<td>-4</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>-2</td>
<td>1</td>
<td>1</td>
<td>-4</td>
<td>1</td>
<td>-4</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>-4</td>
<td>1</td>
<td>-4</td>
<td>1</td>
<td>-4</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>-2</td>
<td>1</td>
<td>1</td>
<td>-4</td>
<td>1</td>
<td>-4</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>-4</td>
<td>1</td>
<td>-4</td>
<td>1</td>
<td>-4</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>-2</td>
<td>1</td>
<td>1</td>
<td>-4</td>
<td>1</td>
<td>-4</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>-4</td>
<td>1</td>
<td>-4</td>
<td>1</td>
<td>-4</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>-2</td>
<td>1</td>
<td>1</td>
<td>-4</td>
<td>1</td>
<td>-4</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

(3) Prove that \( A(k, k+j) = A(k, k+j) = 2^j \) (see also Exercises 14 p. 261, 15 and 16 p. 294).

35. Fermat matrices. Let \( F_n \) be the \( n \)-th section of the Fermat matrix \( F \) composed of the binomial coefficients \( (a, b) = \binom{a+b}{a} \), in the symmetric
notation of p. 8, 0 ≤ a, b ≤ n. So:

\[
F_0 = (1), \quad F_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}, \\
F_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}, \ldots.
\]

Prove that \( F = \mathbf{P} \cdot \mathbf{P}^T \), where \( \mathbf{P} \) is the Pascal matrix (p. 143) and \( \mathbf{P}^T \) its transpose. (2) So, det(\( F_n \)) = 1 (cf. Exercise 46, p. 92) and all coefficients of \( F_n^{-1} \) are integers: \( f_n(i, j) = (-1)^{i+j} \sum_{k \leq n} \binom{i}{j} \binom{k}{j} \). (3) The unsigned coefficients \( C_n(i, j) := \lfloor f_n(i, j) \rfloor \) satisfy: \( C_n(i, j) = C_{n-1}(i-1, j-1) + C_{n-1}(i-1, j) + C_{n-1}(i, j-1) + C_{n-1}(i, j) \), with \( C_n(i, j) = 0 \) if \( i < 0 \) or \( j < 0 \), except \( C_n(-1, -1) = 1 \).

\[
F_0 = (1), \quad F_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 3 & 3 & 1 \\ 1 & 2 & 1 \end{pmatrix},
\[
F_3 = \begin{pmatrix} 4 & 6 & 4 & 1 \\ 6 & 14 & 11 & 3 \\ 4 & 11 & 10 & 3 \\ 1 & 3 & 3 & 1 \end{pmatrix}, \quad C_4 = \begin{pmatrix} 5 & 10 & 10 & 5 & 1 \\ 10 & 30 & 35 & 19 & 4 \\ 10 & 35 & 46 & 27 & 6 \\ 5 & 19 & 27 & 17 & 4 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}, \ldots
\]

(4) \( C_n(k, 0) = C_n(0, k) = \binom{n+1}{k+1}, \quad C_n(k, 1) = \binom{n+1}{k+1} \left( (k+1) \left( (n+1) - 1 \right) \right) / (k+2), \ldots \), and \( \sum_{i, j} C_n(i, j) = (4^{n+1} - 1)/3 \).

**36. Simple and double summations.** Prove the equality ([Carlitz, 1968a]):

\[
\sum_{\substack{i+j+k=n \leq n \leq n}} \binom{i+j+k}{j} \cdot \binom{k+i}{k} = \sum_{0 \leq i \leq n} \binom{2i}{i}.
\]

**37. Two multiple summations.** (1) The summation \( \sum (x_1 x_2 \cdots x_i)^{-1} \) taken over all systems of integers \( x_i \geq 1, i \in \mathbb{N} \) such that \( x_1 + x_2 + \cdots + x_i = n \), equals \( (1/n!) \varepsilon(n, l) \), where \( \varepsilon(n, l) \) is the Stirling number of the first kind, [5d] (p. 213). (2) The summation \( \sum (x_1^n x_2^n \cdots x_i^n) := a_{i, n}(p) \), taken over all systems of integers \( x_i \geq 0, \) such that \( x_1 + x_2 + \cdots + x_i = p \), equals \( \frac{1}{n!} \sum_{k \leq p} k! S(n, k) \left( \begin{pmatrix} p+1-k \choose p-k \right) \right) \), where \( S(n, k) \) is the Stirling number of the second kind, [14s] (p. 51). [Hint: Consider \( \sum_{p \geq 0} a_{i, n}(p) t^p \).]

**38. The formula of Li Jen-Shu** (see, for instance, [Kauczyk, 1964]):

\[
\sum_{0 \leq i \leq n} \binom{k}{i} \left( \frac{n + 2k - j}{2k} \right) = \binom{n + k}{k}^2.
\]

**39. A formula of Riordan** ([Riordan, 1962a], [Gould, 1963a]):

\[
\sum_{0 \leq i \leq n-1} \binom{n-1}{k} n^{n-1-k} (k+1)! = n^n.
\]

**40. A formula of Gould.** If we put \( A_k(a, b) := a(a + bk)^{-1} \left( \frac{a + bk}{k} \right) \), then we have:

\[
\sum_{0 \leq i \leq n-1} A_k(a, b) A_{n-k}(a', b) = A_n(a + a', b).
\]

([Gould, Kauczyk, 1966], and for a 'combinatorial' proof, [Blackwell, Dubins, 1966]. We already met similar numbers in [9b], p. 24.)

*41. The 'Master Theorem' of MacMahon.* The \( a_{r, n}, r, \in \mathbb{N} \) being constants (complex, for instance), let us consider the \( n \) linear forms:

\[
X_r := \sum_{s=1}^{n} a_{r, s} x_s, \quad r \in \mathbb{N}.
\]

The 'Master Theorem' asserts that the coefficient of the monomial \( x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \) (where \( m_1, m_2, \ldots, m_n \) are integers \( \geq 0 \)) in the polynomial \( X_1^{m_1} X_2^{m_2} \cdots X_n^{m_n} \) is equal to the coefficient of the same monomial in \( D^{-1} \), where \( D \) is the determinant:

\[
D := \begin{vmatrix} 1 - a_{11} x_1 & -a_{12} x_2 & \cdots & -a_{1n} x_n \\ -a_{21} x_1 & 1 - a_{22} x_2 & \cdots & -a_{2n} x_n \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} x_1 & -a_{n2} x_2 & \cdots & 1 - a_{nn} x_n \end{vmatrix}
\]

In other words, if the identity matrix is denoted \( I \), if \( A \) is \( [a_{r, s}]_{r, s \in \mathbb{N}} \), if the column matrix of the \( x_r, r \in \mathbb{N} \), is \( X \), if the diagonal matrix
of the $x_i$ is $X$, then we have (with the notation [8a], p. 148):

$$\sum_{i=1}^{n} (AX)^{-1} i = \sum_{i=1}^{n} \frac{1}{(det(I - XA))^{-1}}. $$

([*MacMahon, I, 1915], p. 93. See [Foata, 1964, 1965], [*Cartier, Foata, 1969], pp. 54–60, for a noncommutative generalization, [Good, 1962], from whom we borrow the proof, and [Wilf, 1968b]). [Hint: Put $Y_i = 1 + X_i$, then the required coefficient is equal to the coefficient of $x_i^{n+1} \cdots x_n^{n+1}$ in $Y_1 \cdots Y_n$, hence, by the Cauchy theorem:

$$(2\pi i)^{-n} \int \cdots \int \frac{Y_1 \cdots Y_n}{x_1^{n+1} \cdots x_n^{n+1}} dx_1 \cdots dx_n,$$

where the integration contours are circles around the origin. Then perform the change of variable $w_r := x_r/Y_r$, $r \in [n]$, whose Jacobian causes $D$ to appear.]

42. Dixon formula. This famous identity can be stated as follows:

$$\sum_{s=0}^{2m} (-1)^s \binom{2m}{s}^3 = (-1)^m \binom{3m}{m}!.$$

This is a special case ($a=b=c=m$) of:

$$S := \sum_{s} (-1)^s \binom{b+c}{b+s} \binom{c+a}{c+s} \binom{a+b}{a+s} = \frac{(a+b+c)!}{a!b!c!}.$$

[Hint: Observe that $S = (-1)^{a+b+c} \sum_{x_1, x_2, \ldots} (y-z)^{a+b+c}(z-x)^{a+b+c}(x-y)^{a+b+c}$, and apply then the ‘Master Theorem’ of Exercise 41.]

43. A beautiful identity concerning the exponential. Show that:

$$\exp \left\{ \sum_{m \geq 1} \frac{m^{n-1} f^m}{m!} \right\} = 1 + \sum_{n \geq 1} \frac{(n+1)^{n-1} f^n}{n!}.$$

44. The number of terms in the derivatives of implicit functions ([Comtet, 1974]) The number $a(n)$ of different monomials $A^m_{f_1, f_2, \ldots}$ in the expression of $y = f^{(n)}(x)$, where $f(x, y) = 0$ (see p. 153) is such that

$$a(n) = \frac{1}{m^{n-1}} \prod_{(i,j) \in E} (1 - t_{i,j}),$$

with $E = \mathbb{N} \setminus \{(0, 0), (0, 1)\}$. The first values of $a(n)$ are:

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a(n)$</td>
<td>1</td>
<td>3</td>
<td>9</td>
<td>24</td>
<td>61</td>
<td>145</td>
<td>333</td>
<td>732</td>
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<td>6583</td>
<td>13047</td>
<td>25379</td>
<td>50009</td>
<td></td>
</tr>
</tbody>
</table>

45. Some expansions related to the derivatives of the gamma function. In the sequel, we write $\gamma = 0, 577\ldots$ for the Euler constant, $\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$ (see Exercise 36, p. 88), $\zeta(s, a) = \sum_{n \geq 1} \frac{(a+n)^{-s}}{n}$, $x_k = (-1)^k (k-1)! \zeta(k)$, and $y_n$ for the Bell polynomial, [3c] p. 134. (1) We have:

$$t \Gamma(t) = \Gamma(1 + t) = \exp \left\{ -yt + \zeta(2) t^2/2 - \zeta(3) t^3/3 + \ldots \right\}.$$

Consequently,

$$\Gamma^{(n)}(1) = y_n (-\gamma, x_2, x_3, \ldots) = \int_0^\infty e^{-x} \log^n x \, dx,$$

(2) Hence,

$$\frac{1}{\Gamma(t)} = \sum_{n=0}^{t-1} \frac{r^t}{n!} y_n (y_n - x_2, -x_3, \ldots)$$

(3) Find similar expansions for $\Gamma(a + t)$ using $\zeta(s, a)$.
This chapter solves the following problem: let be given a system \((A_1, A_2, \ldots, A_p)\) of \(p\) subsets of a set \(N\), whose mutual relations are somehow known, compute the cardinal of each subset of \(N\) that can be formed by taking intersections and unions of the given subsets or their complements.

In the sequel, we will denote the intersection of \(A\) and \(B\) by \(AB\) as well as by \(A \cap B\), similarly the complement of \(A\) by \(\bar{A}\) or \(\complement A\). Each subset of \([p] := \{1, 2, \ldots, p\}\) will be denoted by a lower case Greek letter.

**4.1. Number of elements of a union or intersection**

We want to generalize the following formula:

\[ |A \cup B| = |A| + |B| - |AB|, \quad AB := A \cap B, \]

where \(A, B\) are subsets of \(N\), and that follows (notations \([10a]\), p. 25, and \([10d]\), p. 28) from:

\[ A \cup B = A + (B - AB) \Rightarrow |A \cup B| = |A| + |B| - |AB| = |A| + |B| - |AB|. \]

The interpretation of \([1a]\) in Figure 33 is intuitively clear.

\[ \text{Fig. 33.} \]

**Theorem A (Sieve formula, or inclusion-exclusion principle).** Let \(\mathcal{A}\) be a \(p\)-system of \(N\), in other words a sequence of \(p\) subsets \(A_1, A_2, \ldots, A_p\) of \(N\), among which some may be empty or coinciding with each other. Then:

\[ |A_1 \cup A_2 \cup \ldots \cup A_p| = \sum_{1 \leq i \leq p} |A_i| - \sum_{1 \leq i < j \leq p} |A_i A_j| + \sum_{1 \leq i < j < k \leq p} |A_i A_j A_k| - \cdots + (-1)^{p-1} |A_1 A_2 \cdots A_p|. \]

(Formula \([1b]\) is also known as formula of [Da Silva, 1854], [Sylvester, 1883]; it holds whether \(N\) is finite or not.)

First, we indicate two other ways, \([1d, f]\), to write \([1b]\):

1. Using Exercise 9 (p. 158) for (\*) and introducing

\[ S_k := \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq p} |A_{i_1} A_{i_2} \cdots A_{i_k}| := \sum_{x \in \Psi'(p)} |A_{i_1} A_{i_2} \cdots A_{i_k}|, \]

formula \([1b]\) becomes:

\[ |A_1 \cup A_2 \cup \ldots \cup A_p| = \sum_{1 \leq i \leq p} (-1)^{i-1} S_i = \sum_{1 \leq i \leq p} (-1)^{i-1} S_i = S_1 - S_2 + S_3 - \cdots + (-1)^{p-1} S_p. \]

2. Let \(x\) be a subset of \([p] := \{1, 2, \ldots, p\}\), \(x \subset [p]\). We introduce the following notations:

\[ A_x := \bigcap_{i \in x} A_i, \quad A_x = \bigcap_{i \in \complement x} A_i = N, \quad \bigcup_{i \in \complement x} A_i = \emptyset. \]

Formula \([1b]\) becomes (with \(\Psi'(p) := \Psi([p]) = \text{the set of blocks} = \text{the set of nonempty subsets of } [p]\)):

\[ |A_1 \cup A_2 \cup \ldots \cup A_p| = \sum_{x \in \Psi'(p)} (-1)^{|x|-1} |A_x|. \]

We argue by induction on \(p\). Because of \([1a]\) for equality (\*), we get:

\[ |\bigcup_{1 \leq i \leq p+1} A_i| = |A_{p+1} \cup \bigcup_{1 \leq i \leq p} A_i| \]

\[ = |A_{p+1}| + \big| \bigcup_{1 \leq i \leq p} A_i \big| - \bigg| \bigcup_{1 \leq i \leq p} (A_{p+1} A_i) \bigg|, \]

where, if \([1f]\) is supposed to hold, we have (using the notation \(\Psi_2(p) := \{x \mid x \subset [p], |x| \geq 2\}\)):

\[ |\bigcup_{1 \leq i \leq p} A_i| = \sum_{1 \leq i \leq p} |A_i| + \sum_{x \in \Psi_2(p)} (-1)^{|x|-1} |A_x|. \]
\[ [1] \quad \bigcup_{i \in [p]} (A_{p+1} A_i) = \sum_{i \in [p]} \sum_{x \neq 2(i)} (-1)^{|x| - 1} |A_x|. \]

Substituting \([1h, i]\) into \([lg]\) gives then:

\[ P_{k+1} P_k \quad \sum_{i=1}^{p+1} |A_i| = \sum_{k=0}^{|A|} (-1)^{|x| - 1} |A_x|. \]

This formula allows us to compute theoretically \(\pi(n)\) if we know all prime numbers \(\leq \sqrt{n}\).

(2) Chromatic Polynomials. Let \(G \subseteq [n]\) be a graph on the set \((\text{of nodes})\) \([n] = \{1, 2, \ldots, n\}\), and let \(\lambda\) be an integer \(\geq 0\). The chromatic polynomial of \(G\) is the number \(P_\lambda(\lambda)\) of ways to colour the nodes in \(\lambda\) (or fewer) colours such that two adjacent nodes have different colours. Indeed, any colouring is a map of \([n]\) into \([\lambda]\), say \(f : [n] \mapsto [\lambda]\), such that \(i, j \in E \Rightarrow f(i) \neq f(j)\).

For instance, if \(G = \{[1, 2], [2, 3], [3, 4], \ldots, [n-1, n]\}\), we find \(P_\lambda(\lambda) = (\lambda - 1)^{n-1}\) by successively choosing the colours of the nodes \([1, 2, 3, \ldots, n]\). In the same manner, if \(G = [n]\), we find \(P_\lambda(\lambda) = (\lambda - 1)^{n-1}\). Evidently, \(P_\lambda(0) = P_\lambda(1) = 0\). Let us prove that \(P_\lambda(\lambda)\) is a polynomial in \(\lambda\). For each edge \(E \in G\), \(1 \leq i \leq y : = |\{i\} \leq \alpha(<\lambda)|\), let \(A_{[\lambda]} = \{1, 2, \ldots, \lambda\}\) be the set of colourings which give the same colour to the two nodes of \(E_j\). Then, with \([1f]\), \(P_\lambda(\lambda) = |\bar{A}_1 \bar{A}_2 \ldots \bar{A}_y| = \lambda^n - (|A_1| + |A_2| + \cdots + |A_y| + \cdots) + (|A_1 A_2| + |A_1 A_3| + \cdots + \cdots) + \cdots - \cdots \). Now, \(|A_1| = |A_2| = \cdots = \lambda^{n-1}, |A_1 A_2| = |A_1 A_3| = \cdots = \lambda^{n-2}, \ldots\), and any other \(|A_1 A_2 \ldots A_k|, k \geq 3\), is a polynomial in \(\lambda\) with degree \(\leq n-2\), as can be seen easily. Consequently, \(P_\lambda(\lambda) = \lambda^n - g \lambda^{n-1} + a_1 \lambda^{n-2} - a_2 \lambda^{n-3} + \cdots + (-1)^{n-1} a_{n-1}\lambda + \ldots\), where the \(a_i\) are integers, which can all be proven to be \(>0\).

The following pretty results are worthwhile: (I) if the graph \(G\) has connected components \(G_1, G_2, \ldots, G_r\), then \(P_\lambda(G) = P_{G_1}(\lambda) P_{G_2}(\lambda) \cdots P_{G_r}(\lambda)\). (II) \(G\) is a tree if and only if \(P_\lambda(G) = (\lambda - 1)^{n-1}\). (III) \(G\) is a polygon (i.e. circuit), then \(P_\lambda(G) = (\lambda - 1)^{n-1} + (-1)^{n-1}(\lambda - 1)\). (IV) \(G\) is the complete bipartite graph with parts \(M\) and \(N\) (i.e. \(x, y \in G \Rightarrow x \neq y \in M, y \in N\)), then \(P_\lambda(G) = n^2 S(n, 1) (\lambda - 1)^{n-1}\) (see p. 204). (V) \(G\) is connected, then \(P_\lambda(G) = (\lambda - 1)^{n-1}\) for every integer \(1 \leq \lambda \leq \lambda_0\). (VI) The smallest number \(r\) such that \(G\) has a nonzero coefficient in \(P_{\lambda}(\lambda)\) is the number of components of \(G\). (See, for instance, the introductory survey of [Read, 1968].)

Finally, let us mention as still unsolved problems: (I) the characterization of chromatic polynomials; (II) the unimodality (p. 269) of the coefficients \(1, a_2, a_3, a_4, \ldots\); (III) the condition for two graphs to have the same chromatic polynomial.

**Definition.** A system \((A_1, A_2, \ldots, A_p)\) of subsets of \(N\) is called interchangeable if and only if the cardinality of any intersection of \(k\)
arbitrary subsets among them depends only on k, for all k \in [p].

**Theorem C.** Let be given an interchangeable system of subsets of \( N \), say \((A_1, A_2, \ldots, A_p)\); then we have:

\[
\prod_{1 \leq i < j \leq p} (|A_i| + |A_j| - 1) = \sum_{0 \leq k \leq p} (-1)^{p-k} \binom{p}{k} |A_k|.
\]

This is an immediate consequence of the definition of interchangeable systems and of [1b, j].

### 4.2. The 'Problème des Rencontres'

**Definition.** A permutation (Definition B, p. 7) \( \sigma \) of \( N, |N|=n \), is called a derangement, if it does not have a fixed point, or rencontre, or coincidence, in the sense that for all \( x \in N \), \( \sigma(x) \neq x \).

For example, the permutation \( \sigma_1 := (abcde) \) does not have a coincidence, while \( \sigma_2 := (cdeab) \) has 2. The famous 'problème des rencontres' ([*Montmort, 1708*]) consists of computing the number \( d(n) \) of derangements of \( N, n=|N| \).

**Theorem A.** The number \( d(n) \) of derangements of \( N, n=|N| \), equals:

\[
d(n) = \sum_{0 \leq k \leq n} (-1)^k \frac{n!}{k!} = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}\right)
\]

or also, for \( n \geq 1 \), the integer closest to \( n! e^{-1} \):

\[
d(n) = \|n! e^{-1}\|
\]

(Because of [2a], Chrystal has suggested the name \( n \) antifactorial for \( d(n) \), and the notation \( n! \).)

If we identify \( N \) with \([n] := \{1, 2, \ldots, n\}\), we denote the set of permutations of \([n]\) by \( \mathcal{S}[n] \), and the subset of \( \mathcal{S}[n] \) consisting of permutations \( \sigma \) such that \( \sigma(i) \neq i \), \( i \in [n] \), by \( \mathcal{S}_i = \mathcal{S}[i] \), and the set of derangements of \([n]\) by \( \mathcal{D}[n] \). Clearly \( \mathcal{S}[n] = \mathcal{D}[n] + \bigcup_{i=1}^n \mathcal{S}_i \). Hence, by Theorem B (p. 7), for (̊):

\[
\|n! e^{-1}\| |\mathcal{S}[n]| = d(n) + \bigcup_{0 \leq i \leq n} \mathcal{S}_i.
\]

Now the \( \mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n \) are interchangeable (Definition p. 179), since giving a \( \sigma \in \mathcal{S}_1 \mathcal{S}_2 \ldots \mathcal{S}_n \) is equivalent to giving one of the permutations of \([n] - \{i_1, i_2, \ldots, i_k\}\), whose total number is \((n-k)! \) \( (i_1 < i_2 < \cdots < i_k) \). Thus, [2a] follows from [11] applied to \( |\bigcup_{i=1}^n \mathcal{S}_i| \) in [2b]. Finally, for [2a'], use in (̊) the well-known inequality that relates the rest of an alternating series to the first neglected term:

\[
\|n! e^{-1} - d(n)\| = n! \left| \sum_{q=1}^n \frac{(-1)^{q-1}}{q!} \right| < n! \frac{1}{(n+1)!} = \frac{1}{n+1} \leq \frac{1}{2}.
\]

In particular, [2a] shows that \( \lim_{n \to \infty} \{d(n)/n!\} = 1/e \). The way the number \( e \) intrudes here into a combinatorial problem has strongly appealed to the imagination of the geometers of the 18-th century. In more colourful terms, if the guests to a party leave their hats on hooks in the cloakroom, and grab at good luck a hat when leaving, then the probability that nobody gets back his own hat is (approximately) \( 1/e \).

Another method of computing \( d(n) \) consists of observing that the set \( \mathcal{S}_K[n] \) of permutations of \([n]\) for which \( K(\subset [n]) \) is the set of fixed points, has for cardinality \( d(n-K) \). So:

\[
\mathcal{S}[n] = \sum_{K \subset [n]} \mathcal{S}_K[n] = \sum_{k=0}^n \left( \sum_{|K|=k} \mathcal{S}_K[n] \right).
\]
Hence \( n! = |S[n]| = \sum_{k=0}^{n} \binom{n}{k} d(n-k) = \sum_{h=0}^{n} \binom{n}{h} d(h) \), from which

**Theorem B.** The number \( d(n) \) of derangements of \([n]\) has for generating function:

\[
\varphi(t) := \sum_{n \geq 0} d(n) \frac{t^n}{n!} = e^{-t} (1-t)^{-1}.
\]

In fact, using [2a] for (**):\[
\sum_{n \geq 0} d(n) \frac{t^n}{n!} = \sum_{n \geq 0} \binom{n}{k} \frac{(-1)^k}{k!} = \left( \sum_{k \geq 0} \frac{t^k}{k!} \right)^n = (1 - t)^n,
\]

- Taking the derivative of \( e^{-t} (1-t)^{-1} \), we get \(-e^{-t} (**) - \varphi + (1-t) \varphi' (**) - (1-t) \varphi \), and then we equate coefficients in (**') to obtain [2d], and in (**') to obtain [2d'] (combinatorial proofs are also easy to find).\]

**Theorem C.** The number \( d(n) \) of derangements of \([n]\) satisfies the following recurrence relations:

\[
\begin{align*}
[2d] & \quad d(n+1) = (n+1) d(n) + (-1)^n + 1; \\
[2d'] & \quad d(n+1) = n d(n) + d(n-1).
\end{align*}
\]

We discuss now a natural generalization of the 'problème des rencontres'. A \((k \times n)\)-latin rectangle will be any rectangular matrix with \(k\) rows and \(n\) columns consisting of integers \(\in [n]\), and such that all integers occurring in any one given row or column are all different \((k \leq n)\). We suppose that the first row is \(\{1, 2, 3, \ldots, n\}\) in this order (and we say that the rectangle is **reduced** then). We give an example of a \((3 \times 5)\) latin rectangle:

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
3 & 1 & 4 & 5 & 2 \\
5 & 3 & 1 & 2 & 4
\end{pmatrix}
\]

The number \(K_n\) of (reduced latin) \((3 \times n)\)-rectangles satisfies several recurrence relations (see, for instance, [Jacob, 1930], [Kerawala, 1941], [*Riordan, 1958], p. 204) and today there are asymptotic expansions known for it ([*Riordan, 1958], p. 209). The first values are (taken from tables of Kerawala, \(n \leq 15\)):

\[
\begin{array}{c|cccccccccccccccccccc}
\hline
\hline
K_n & 2 & 24 & 552 & 21280 & 1073760 & 70299264 & 486582256 & 3701523200 & 29662418592 & 238474298688 & 1949531348544 & 16100320458352 & 131910669034400 & 1086441940183040 & 8927821138402480 & 74496849938440096 & 633649135587208896 & 5440298309918810496 \\
\hline
\end{array}
\]

We see that \(K_n\) does not satisfy any known recurrence relations.\]

**4.3. The ‘Problème des Ménages’**

This is the following problem: **What is the number of possible ways one can arrange \(n\) married couples (=ménages) around a table such that men and women alternate, but no woman sits next to her husband.** (Posed, solved and popularized by [*Lucas, 1891*]. See also [Cayley, 1878a, b]; [Moser, 1967] gives an interesting generalization.)

We suppose the wives already placed around the table (2, \(n!\) pos-
sibilities). We number them 1, 2, ..., n in the ordinary (counterclockwise) direction, starting from one of them: $E_1, E_2, ..., E_n$ (Figure 34, $n=6$). We assign to every husband the number of his wife: $M_1, M_2, ..., M_n$, and to every empty seat the number of the wife to the right: $S_1, S_2, ..., S_n$. The problem consists of counting the number of possible admissible assignments of seats to husbands. Such an assignment is tantamount to giving a permutation $\sigma$ of $[n] = \{1, 2, ..., n\}$, where $\sigma(i)$ stands for the seat number assigned to husband $M_i$, in $[H]$. This number should satisfy:

$$[3a] \quad \sigma(i) \neq i, \quad \sigma(i) \neq i + 1 \quad \text{for} \quad i \in [n-1],$$

$$\sigma(n) \neq n, \quad \sigma(n) \neq 1.$$

Let $\mu(n)$ be the number of permutations such that $[3a]$ holds; this is usually called the 'reduced number of ménages'. The total number $\mu(n)$ of placements of ménages is hence equal to $2\cdot n! \mu(n)$, if we take into account the $2\cdot n!$ possibilities of arranging the wives. We concentrate now on computing $\mu(n)$. The main idea consists of connecting this problem with the theorem on p. 24. To carry this out, we put:

$$[3b] \quad A_{2i-1} := \{\sigma \mid \sigma(i) = i\}, \quad i \in [n];$$

$$A_{2i} := \{\sigma \mid \sigma(i) = i + 1\}, \quad i \in [n-1];$$

$$A_{2n} := \{\sigma \mid \sigma(n) = 1\}.$$

Clearly, by [3a, b] for ($\ast$), and [1] (p. 178), for ($\ast\ast$):

$$[3c] \quad \mu(n) = \sum_{i=1}^{2n} (-1)^{i} |A_i| \sum_{\beta \subseteq [2n]} (-1)^{|\beta|} |A_\beta|.$$

Now, $|A_\beta| = |\bigcap_{i \in \beta} A_i|$ is evidently equal to 0 if $\beta$ contains two consecutive elements of the 'circle' (1, 2, 3, ..., 2n, 1). In the opposite case, $|A_\beta|$ equals $(n-|\beta|)!$ and, according to the Theorem on p. 24, such $\beta$ happens $g_1(2n, k)$ times; hence:

$$\mu(n) = \sum_{k=0}^{n} (-1)^{k} (n-k)! g_1(2n, k).$$

Finally, we obtain:

**Theorem.** The number $\mu(n)$ of reduced solutions to the 'ménages' problem, defined above, equals:

$$[3d] \quad \mu(n) = \sum_{0 \leq k \leq n} (-1)^{k} \frac{2n!}{2n-k} \binom{2n-k}{n-k}.\n$$

This beautiful formula (due to [Touchard, 1953]) is perhaps not the best for the actual computation of the $\mu(n)$: several recurrence relations for more efficient computations are known. (See *[Riordan, 1958], pp. 195-201, [Carlitz, 1952a, 1954a], [Gilbert, 1956a], [Kaplansky, Riordan, 1946], [Kerawala, 1947b], [Riordan, 1952a], [Schöbe, 1943, 1961], [Touchard, 1943].) The first values of $\mu(n)$ (taken from the tables of [Moser, Wyman, 1958a], $n \leq 65$, are:

<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu(n)$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>13</td>
<td>80</td>
<td>579</td>
<td>4738</td>
<td>43387</td>
</tr>
<tr>
<td>$n$</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu(n)$</td>
<td>439792</td>
<td>4890741</td>
<td>59216642</td>
<td>775569313</td>
<td>10927434464</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4.4. **Boolean algebra generated by a system of subsets**

Let $\mathcal{A} = (A_1, A_2, ..., A_p)$ be a system of subsets of a set $N$, $A_i \subseteq N$, $i \in [p]$, among which there may be identical or empty subsets.

**Definition A.** The Boolean algebra (of subsets) generated by $\mathcal{A}$, denoted by $b(\mathcal{A})$, is the set of subsets of $\mathcal{A}$ that can be obtained by means of a finite number of the set operations: union, intersection and complementation. Each of the elements of $b(\mathcal{A})$ will be called Boolean function generated by $\mathcal{A}$.

It can be immediately verified that, for the operations $\cap$, $\cup$ and...
$W \rightarrow \mathcal{W}$, $b(\mathcal{A})$ is actually a Boolean algebra in the sense of p. 2. The following are two examples of Boolean functions generated by $(A_1, A_2, A_3)$ (we recall that the notation $ST$ means $S \cap T$):

\[ f_1 = (A_1A_2) \cup \sim A_1, \quad f_2 = A_1 \cup (\sim A_1 \cup A_2) A_3. \]

As for polynomials, it is sometimes very interesting to interpret any Boolean function $f \in b(\mathcal{A})$ as a purely formal expression of the 'variables' $(A_1, A_2, ..., A_n)$ and to introduce an equivalence relation on the set of these expressions by putting $f \sim g$ when $g$ can be obtained from $f$ by the rules of computation in any Boolean algebra (see p. 2). For example, $f := (\sim A_1 \cup \sim A_2 \sim A_3) = (A_1 \cup A_2 A_3)$ is true, but $f := A_1A_2 \sim A_1 \cup A_2$ is not true.

**Definition B.** The complete products of $\mathcal{A}$ are the $2^n$ Boolean functions of the form (see notation [1e], p. 177):

\[ A_1A_2 = (\cap A_i) \cap (\cap \sim A_i), \text{ where } \alpha \in [p]. \]

The set of complete products is called $\mathcal{A}(\mathcal{A})$.

For instance, the 8 complete products of $\mathcal{A} = (A_1, A_2, A_3)$ are:

\[ A_1A_2A_3, \quad A_1A_3A_3, \quad A_1A_2A_3, \quad A_1A_2A_3, \]
\[ A_1A_2A_3, \quad A_1A_2A_3, \quad A_1A_2A_3. \]

**Definition C.** The conjunctions of $\mathcal{A}$ are the $2^n$ Boolean functions of the form:

\[ A_1 : = \cap A_i, \text{ where } \lambda \in [p]. \]

The set of conjunctions can be denoted by $c(\mathcal{A})$.

For instance, the 8 conjunctions of $\mathcal{A} = (A_1, A_2, A_3)$ are $N, A_1, A_2, A_3, A_1A_2, A_2A_3, A_3A_1, A_4A_2A_3$.

**Theorem A.** Each Boolean function has a unique representation as a union of complete products (up to order). Hence (with the notation $\sum$ of [10a] of p. 25 for the disjoint union):

\[ \forall f \in b(\mathcal{A}), \exists! \mathcal{A} \subset b(\mathcal{A}) \text{ such that } f = \sum_{\mathcal{A} \in \mathcal{A}} M. \]

We say in this case that $f$ is put in the canonical disjunctive form.

From this theorem it follows that there are $2^n$ different (non equivalent) Boolean functions in $b(\mathcal{A})$. We give a sketch of proof of the theorem.

(1) The proposition is evidently true for all $A_i \in \mathcal{A}$, because $A_i = \sum_{i \in \mathcal{A}} A_i \sim A_i$.

(2) If $f, g \in b(\mathcal{A})$ are brought into the canonical disjunctive form, then $f \cup g$ can be brought into canonical form too, because for $f = \bigcup_{i \in \mathcal{A}} B_i, g = \bigcup_{j \in \mathcal{A}} C'_j$, where $\mathcal{A}, \mathcal{A}' \subset b(\mathcal{A})$, we have $f \cup g = \bigcup_{i \in \mathcal{A}} B_i \cup C'_j$.

(3) Similarly, for

\[ f \cap g = (\bigcup_{i \in \mathcal{A}} B_i) \cap (\bigcup_{j \in \mathcal{A}} C'_j) = \bigcup_{i \in \mathcal{A}} B_i \cap C'_j, \]

by means of [1g] (p. 3), for (2).

(4) Finally, for the passage to the complement, we have:

\[ f = \bigcup_{i \in \mathcal{A}} B_i \cap C'_j \bigcap B_i \cap C'_j, \]

with [1e] (p. 3), for (2).

(1), (2), (3), (4) make it hence possible to reduce any $f \in b(\mathcal{A})$ step by step.

By way of example, we show the reduction of the functions [4a]:

\[ f_1 = A_1A_2 \cup \sim A_3 = (A_1A_2A_3 \cup A_1A_2A_3) \cup \]
\[ \cup (A_1A_2A_3 \cup A_1A_2A_3 \cup A_1A_2A_3 \cup A_1A_2A_3), \]
\[ = A_1A_2A_3 + A_1A_2A_3 + A_1A_2A_3 + A_1A_2A_3, \]
\[ f_2 = A_1 \cup (A_1 \cup A_2) A_3 = A_1 \cup \bigcup_{i \in \mathcal{A}} (A_1 \cup A_2) \cup \sim A_3, \]
\[ = A_1 \cup A_2 \cup \sim A_3 = \bigcup_{i \in \mathcal{A}} A_1A_2A_3 = A_1A_2A_3 + A_1A_2A_3 + A_1A_2A_3 + A_1A_2A_3. \]

We have already met, on pp. 25 and 28, in the set $\mathcal{P}(N)$ of subsets of $N$, the operations $+$ and $-$, whose definition we recall now. For $A, B, C, D \subset N$, we put:

\[ 4f \quad C = A + B \iff C = A \cup B, \quad A \cap B = \emptyset \]
\[ 4g \quad D = A - B \iff A = B + D \iff D = A \setminus B, B \subset A. \]
It follows then for the cardinalities:

\[ |A + B| = |A| + |B|, \quad |A - B| = |A| - |B| \]

and for the rules of computation:

\[ |I| \quad (I) \quad (A + B) \cdot C = A + (B \cdot C). \]
\[ (II) \quad A + B = B + A. \]
\[ (III) \quad A + \emptyset = \emptyset + A = A. \]
\[ (IV) \quad A + \overline{A} = \emptyset. \]
\[ (V) \quad A \cdot (B + C) = AB + AC. \]
\[ (VI) \quad A - \emptyset = A. \]
\[ (VII) \quad A - (B - C) = (A - B) + C \quad \text{(provided the two pairs of brackets make sense according to } [4g]). \]

**Theorem B.** The cardinal number \(|f|\) of every Boolean function \(f \in \mathcal{B}(\mathcal{A})\) can be expressed as a linear combination with integer coefficients \(\leq 0\), of the cardinals of the conjunctions of \(\mathcal{A}^2\):

\[ \forall f \in \mathcal{B}(\mathcal{A}), \quad \exists \{ I_1, I_2, \ldots, I\} \subseteq \mathcal{Z}, \quad \exists \{ C_1, C_2, \ldots, C\} \subseteq \mathcal{A}^2, \quad |f| = \sum_{i \leq j} I_i |C_i|. \]

According to [4e], it suffices to prove [4j] for each complete product \(M\), because \(|f| = \sum_{\mathcal{E}} |M| |\mathcal{E}|\); this fact is proved in the following theorem.

**Theorem C.** Let \(B \in \mathcal{B}(\mathcal{A})\) be a subset of \(N\) which is the intersection of some \(A_i\) and \(\overline{A}_j\):

\[ B = (\bigcap_{i \in \lambda} A_i) \cdot (\bigcap_{j \in \mu} \overline{A}_j), \quad \text{where } \lambda + \mu = [p]. \]

Then, the cardinal \(|B|\) can be computed by performing successively the following operations:

1. Replace in [4k] the \(\overline{A}_j\) by \(1 - A_j\).
2. Expand the new form, thus obtained, of [4k] into a polynomial in the variables \(A_1, A_2, \ldots, A_n\), \(\lambda \subseteq A, \mu \subseteq B\), the \(\cap\) being considered as product operation.

(3) Replace every monomial by its cardinal number and replace the monomial \(1\) (if it occurs) by \(n(=|N|)\).

We illustrate this rule by computing the cardinal of \(P \overline{Q}R:\)

\[ P \overline{Q}R \rightarrow P + (1 \cdot Q) - (1 \cdot R) \rightarrow P \cdot Q + P \cdot R + Q \cdot R \rightarrow \]

\[ |P| + |Q| + |R|, \quad |P\cdot Q\cdot R| = |P\overline{Q}R| \]

So we have \(|W\setminus V| = |V| - |W\setminus V|\); this formula is evident. Then we put \(W := \bigcap_{j \in \mathcal{E}} A_j\) and \(V := \bigcap_{i \in \mathcal{E}} \overline{A}_j\). Then \(|B| = |W\setminus V| = |V| - |W\setminus V\bigcup \bigcup_{j \in \mathcal{E}} A_j| = |V| - |\bigcup_{j \in \mathcal{E}} V \bigcup \bigcup_{j \in \mathcal{E}} A_j|\). In other words, by [1b]:

\[ |B| = |V| - \sum_{j \in \mathcal{E}} |V \setminus A_j| - \sum_{\{j_1, j_2\} \in \mathcal{E}(\mathcal{E})} |V \setminus A_{j_1} \setminus A_{j_2}| - \text{ etc.} \]

So, in example [4a],

\[ f_1 = (A_1 \cup A_2 \cup A_3) \cup (A_1 \cup \overline{A}_2 \cup \overline{A}_3) \cup (A_1 \cup \overline{A}_3) = A_1 A_2 \overline{A}_3 \overline{A}_3, \]

hence \(|f_1| = n - |A_3| + |A_1 A_2 A_3|\). Similarly, \(f_2 = \overline{A}_1 \cdot (A_1 \cup A_2) \cdot A_3 = A_1 \overline{A}_1 \overline{A}_1 \overline{A}_3\); hence, with the example \(P \overline{Q}R\) above, \(|f_2| = n - |A_3| + |A_1 A_2 A_3|\) or \(|f_2| = n - |A_3| + |A_1 A_2 A_3| + |A_1 A_2 A_3|\). (On this section see also [*Loève, 1963*, p. 44.)

### 4.5. The Method of Rényi for Linear Inequalities

**Definition A.** Let \(f\) be a (set) function mapping a certain Boolean algebra of subsets of \(N\), say \(\mathcal{B}\), onto a set of real numbers \(\mathcal{R} = [0, \infty)^N\). We say that \(f\) is a measure on \((N, \mathcal{B})\), and we denote \(f : \mathcal{B} \rightarrow \mathcal{R}\), if and only if \(f\) is additive, in the sense that for each pair \((B_1, B_2)\) of disjoint subsets of \(N (\equiv B_1 + B_2 \subseteq N)\), belonging to \(\mathcal{B}\), we have:

\[ f(B_1 + B_2) = f(B_1) + f(B_2). \]

The triple \((N, \mathcal{B}, f)\) is then called a measure space.

(4.5.1) We say that \(f\) is a measure on \((N, \mathcal{B})\), and we denote \(f : \mathcal{B} \rightarrow \mathcal{R}\), if and only if \(f\) is additive, in the sense that for each pair \((B_1, B_2)\) of disjoint subsets of \(N (\equiv B_1 + B_2 \subseteq N)\), belonging to \(\mathcal{B}\), we have:

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The triple \((N, \mathcal{B}, f)\) is then called a measure space.

(4.5.1) So \(\mathcal{B}\) is a system of subsets of \(N\), containing \(\emptyset\) and \(N\), and closed under the operations of complementation, finite union and finite intersection, \([1d], p. 2).\]

Hence, for each measure \(f\), we have \(f(\emptyset) = 0\), and for all pairwise disjoint \(B_1, B_2, \ldots, B_n \in \mathcal{B}\):

\[ f \left( \bigcup_{i=1}^n B_i \right) = \sum_{i=1}^n f(B_i). \]
DEFINITION B. The measure space \((N, \mathcal{B}, f)\) is said to be a probability space, if \(f(N) = 1\). In this case \(f\) is called a probability measure, or probability, and will often be denoted by \(P\). Each set \(B \in \mathcal{B}\) is called an event. \(N\) is the certain event, mostly denoted \(\Omega\). Each point \(\omega \in \Omega\) is called a sample.

DEFINITION C. An atom of the Boolean algebra \(\mathfrak{b}(\mathcal{A})\) generated by \(\mathcal{A} = (A_1, A_2, \ldots, A_p)\) (Definition A, p. 185) is a nonempty complete product (Definition B, p. 186). We denote the set of atoms of \(\mathfrak{b}(\mathcal{A})\) by \(\mathfrak{a}(\mathcal{A})\).

THEOREM A. A probability measure \(f\) on \(\mathfrak{a}(\mathcal{A})\) is completely determined by the values (\(\geq 0\)) of \(f\) on each atom \(C \in \mathfrak{a}(\mathcal{A})\) (the set of values of \(f\) on the atoms is only subject to the restriction \(\sum_{C \in \mathfrak{a}(\mathcal{A})} f(C) = 1\)).

This follows from the fact that \(\mathfrak{a}(\mathcal{A})\) is a partition of \(N\), and that every subset \(B \in \mathfrak{b}(\mathcal{A})\) is a union of disjoint atoms (Theorem A, p. 186).

THEOREM B. Let \(\mathcal{A} = (A_1, A_2, \ldots, A_p)\) be a system of subsets of \(N\), \(A_i \subseteq N\), \(i \in [p]\), and let \(\mathfrak{M}\) be the set of measures on the Boolean algebra \(\mathfrak{b}(\mathcal{A})\) generated by \(\mathcal{A}\). Let \(\mathfrak{M}^*\) be the subset of \(\mathfrak{M}\) consisting of the measures \(g\) which are zero on all atoms \(C\) of \(\mathfrak{a}(\mathcal{A})\) except one, \(C_0\), called supporting atom, for which \(g(C_0) = 1\). Here, \(C_0\) runs through \(\mathfrak{a}(\mathcal{A})\). Then for every sequence of \(I\) real numbers \((b_1, b_2, \ldots, b_I)\), and every sequence of \(I\) subsets taken from \(\mathfrak{b}(\mathcal{A})\), say \((B_1, B_2, \ldots, B_I)\), the following conditions \([5c]\) and \([5d]\) are equivalent:

\[ [5c] \quad \text{For all } f \in \mathfrak{M}, \quad \sum_{1 \leq k \leq I} b_k f(B_k) \geq 0. \]

\[ [5d] \quad \text{For all } g \in \mathfrak{M}^*, \quad \sum_{1 \leq k < I} b_k g(B_k) \geq 0. \]

([Rényi, 1958] and [*], 1966, pp. 30–33. See also [Galambos, 1966]. For a generalization to certain quadratic and cubic, etc., inequalities, see [Galambos, Rényi, 1968].)

The fact that \([5c]\) implies \([5d]\) follows from the fact that \(\mathfrak{M}^* \subseteq \mathfrak{M}\). Conversely, let \(g \in \mathfrak{M}^*\), so there exists a \(C_0 \in \mathfrak{a}(\mathcal{A})\) such that:

\[ [5c] \quad g(C_0) = 1, \quad \text{and } g(C) = 0 \quad \text{if } C \in \mathfrak{a}(\mathcal{A}), \; C \neq C_0. \]

Now, according to \([5d]\), with \(a = \alpha(\mathcal{A})\) for short, \([5b]\) for (*), a permutation of the summation order for (**), and \([5c]\) for (***):

\[ 0 \leq \sum_{1 \leq k \leq I} b_k g(B_k) = \sum_{1 \leq k \leq I} b_k \left( \sum_{C \in \mathfrak{a}(\mathcal{A})} f(C) \right) \]

\[ \leq \sum_{1 \leq k \leq I} b_k \left( \sum_{C \in \mathfrak{a}(\mathcal{A})} g(C) \right) \]

\[ = \sum_{C \in \mathfrak{a}(\mathcal{A})} g(C) \left( \sum_{C \in \mathfrak{a}(\mathcal{A})} b_k \right) = \sum_{C \in \mathfrak{a}(\mathcal{A})} b_k. \]

Because the measure \(g \in \mathfrak{M}^*\) is arbitrary, it follows that for each atom \(C = C_0\) from above, we have:

\[ [5f] \quad \sum_{C \in B_k} b_k \geq 0. \]

Let us now consider \([5c]\). We can compute by the same way, now using \([5f]\) for (**):

\[ \sum_{1 \leq k \leq I} b_k f(B_k) = \sum_{1 \leq k \leq I} b_k \left( \sum_{C \in \mathfrak{a}(\mathcal{A})} f(C) \right) \]

\[ = \sum_{C \in \mathfrak{a}(\mathcal{A})} f(C) \left( \sum_{C \in \mathfrak{a}(\mathcal{A})} b_k \right) \geq 0. \]

THEOREM C. Notations as in Theorem B. The conditions \([5c]\) and \([5d]\) remain equivalent if all \('\geq 0'\) signs are simultaneous replaced by \('\leq 0'\) or by \('= 0'\).

In the first case, replace the sequence \((b_1, b_2, \ldots, b_I)\) of Theorem B by \((-b_1, -b_2, \ldots, -b_I)\). In the second case, observe that \(x = 0 \Leftrightarrow x \geq 0\) and \(x < 0\).

Examples of applications of Rényi's method follow now.

4.6. POINCARÉ FORMULA

The method of the preceding section will enable us to show very quickly various equalities and inequalities concerning measures \(f\) associated with a finite system \((A_1, A_2, \ldots, A_I)\) of subsets of \(N\).

With every measure \(f\) on \((N, \mathfrak{b}(\mathcal{A}))\) (Definition A, pp. 185 and
189) and every integer \(k \in [p]\) we associate, as in [1c], p. 177 (using the notation [1e], p. 177, for (*)):

\[
S_k = S_k(\mathcal{A}) = S_k(f, \mathcal{A}) = \sum_{\mathcal{A} \in \mathcal{P}[p]} f(A_1, A_2, \ldots, A_k) := \sum_{1 \leq i < j < \ldots < \ell \leq p} f(A_i, A_j, \ldots, A_{\ell}) = \sum_{x \in \mathcal{P}[p]} f(x).
\]

**Theorem.** For every measure \(f \in \mathfrak{M}(N, b(\mathcal{A}))\), where \(\mathcal{A} = (A_1, A_2, \ldots, A_p)\), \(A_i \subset N, i \in [p]\), the \(S_k\) being defined by [6a], we have:

\[
\begin{align*}
S_0 &:= f(N) \\
\sum_{x \in \mathcal{P}[p]} (-1)^{|x|-1} f(x) &:= \sum_{0 \leq i \leq p} (-1)^i S_i \\
f(A_1 A_2 \ldots A_p) &:= \sum_{x \in \mathcal{P}[p]} (-1)^{|x|} f(x) = \sum_{0 \leq i \leq p} (-1)^i S_i.
\end{align*}
\]

In the case that \(f\) is a probability, [6b] is often called the 'Poincaré formula'. If \(f\) stands for the cardinal number function, then \(f(N) = n\), \(f(M) = n/p_1\), \(f(M_1 M_2) = n/p_1 p_2\), from which we obtain, after an evident factorization:

\[
\phi(n) = n \prod_{p} \left(1 - \frac{1}{p}\right).
\]

If we had defined \(f\) by \(f(X) = \sum x \in X x\), where \(X \subset [n]\), then we would have found \(f(M_1) = p_1 + 2p_1 + \ldots + (n/p_1) = n^2/2 + n/2\), \(f(M_1 M_2) = p_1 p_2 + 2p_1 p_2 + \ldots + (n/p_1 p_2) = n^2/2 + n/2\), hence, after simplifications:

\[
f(\Phi) = \sum x \in \mathcal{P}[p] = \phi(n).
\]

### Example: Euler function

For any integer \(n \geq 1\), let \(\Phi = \Phi(n)\) be the set of positive integers \(x\) which do not exceed \(n\), and relatively prime with respect to \(n\), \(1 \leq x \leq n\). The number \(\phi(n) = |\Phi|\) is called the Euler function of \(n\) and we are going to compute it now. Let the decomposition of \(n\) into prime factors be \(n = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}\) and let \(M_i\) be the set of multiples of \(p_i\) which are smaller than or equal to \(n\). Clearly, \(\Phi = M_1 M_2 \ldots M_k\). Hence, for each measure \(f\) on \([n]\), we get by [6c]:

\[
f(\Phi) = f([n]) = \sum_{k=1}^{a_1} f(M_i) + \sum_{i=1}^{a_2} f(M_i M_j) - \ldots.
\]

The first member \(g(A_1, A_2, \ldots) = 1\). On the other hand:

\[
g(A_\lambda) = \begin{cases} 1 & \text{if } \lambda = \lambda \\ 0 & \text{otherwise.} \end{cases}
\]

The second member of [6b] is hence equal to 1, too, since with [6d] for (*):

\[
\sum_{x \in \mathcal{P}[p]} (-1)^{|x|-1} g(x) = \sum_{x \in \mathcal{P}[p]} (-1)^{|x|} = \sum_{1 \leq i \leq k} (-1)^{i-1} \binom{i}{k} = 1 - (1 - 1)^i = 1. \]

### 4.7. Bonferroni Inequalities

**Definition.** Let \(R\) be an alternating sum of \(a_k \geq 0, k \in [r]\):

\[
R = \sum_{1 \leq i \leq r} (-1)^{r-1} a_i = a_1 - a_2 + \ldots + (-1)^{r-1} a_r.
\]

We say that [7a] satisfies the alternating inequalities, if and only if \((-1)^k (R + \sum_{k=1}^{r} (-1)^k a_k) \geq 0\) for all \(k \in [r]\). In other words:

\[
R \leq a_1, \quad R \geq a_1 - a_2, \quad R \leq a_1 - a_2 + a_3, \ldots
\]

**Theorem ([Bonferroni, 1936]).** Let the \(S_k\) be defined by [6a] (p. 192), then for all measures \(f \in \mathfrak{M}(N, b(\mathcal{A}))\), the sum \(\sum_{k=1}^{r} (-1)^k S_k\), introduced in [6b] (p. 192), satisfies the alternating inequalities. Hence, for each
\( k \in [p] \), we have:

\[ [7c] \quad (\{-1\}^k \{f(A_{i} \cup \ldots \cup A_p) + \sum_{1 \leq i \leq k} (-1)^i S_i \} \geq 0. \]

Quite similarly, with \([6c]\), we have

\[ [7c'] \quad (\{-1\}^{k+1} \{f(A_1 \cdots \cdots \cdots \cdots \cdots \cdots A_p) + \sum_{0 \leq i \leq k} (-1)^{i+1} S_i \} \geq 0. \]

Particularly, for \( f(W) = |W| \) is the cardinal of \( W \), we obtain (cf. \([1c]\), p. 177):

\[ [7d] \quad |A_1 \cup \ldots \cup A_p| \leq \sum_{1 \leq i \leq p} |A_i| \quad \text{(Boole inequality)} \]

\[ |A_1 \cup \ldots \cup A_p| \geq \sum_{1 \leq i \leq p} \sum_{1 \leq i \leq p} |A_i| - \sum_{1 \leq i \leq p} |A_i|, \quad \text{etc.} \]

and the analogous inequalities in the case that \( f = \mathbb{P} \) is a probability.

According to Theorem B (p. 190) it suffices to prove \([7c]\) for an arbitrary measure \( g \in \mathfrak{M}^* \). Let \( \lambda \) have the sense given in the proof of the Theorem, on p. 192, then the first member of \([7c]\) is evidently equal to 0 if \( \lambda = 0 \). Otherwise, we get, with \( I = I_{|\lambda|} \geq 1 \), and \([6d]\) (p. 192, where \( x \) is replaced by \( \eta \)) for \((*)\):

\[ g(A_1 \cup A_2 \cup \ldots \cup A_p) + \sum_{1 \leq i \leq k} (-1)^i S_i(g) \]

\[ = g(A_1 \cup \ldots \cup A_p) + \sum_{\eta \in \mathfrak{P}_k} (-1)^{|\eta|} g(A_\eta) \]

\[ \geq \sum_{\eta \in \mathfrak{P}_k} (-1)^{|\eta|} g(A_\eta) \]

\[ = 1 - \left( \frac{1}{1} + \frac{1}{2} \right) + \ldots + (-1)^k \left( \frac{k}{k} \right) := W_k. \]

Now, by applying the Taylor formula of order \( k \) in \( x = 0 \) to the function \((1 - x)^l, k < l - 1 \), we get for all \( x \in \mathbb{R}, 0 < \theta(x) < 1 \):

\[ [7e] \quad (1 - x)^l = 1 - \binom{l}{1} x + \ldots + (-1)^k \binom{l}{k} x^k + \]

\[ + (-1)^{k+1} \binom{l}{k+1} (1 - x \theta(x))^{l-k-1}. \]

If we put \( x = 1 \) in \([7c]\), we find \((-1)^k W_k = \binom{l}{k+1} (1 - \theta(1))^{l-k-1} > 0 \), in other words, \([7c]\) for all \( g \in \mathfrak{M}^* \).

4.8. Formulas of Ch. Jordan

Theorem A (\([\text{Charles Jordan, 1926, 1927, 1934, 1939}]\). Let \( N_\mathcal{A}(\mathcal{A}) \) stand for the set of points of \( N \) that are covered by exactly \( r \) subsets of the system \( \mathcal{A} = (A_1, A_2, \ldots, A_p) \), then we have for every measure \( f \in \mathfrak{M}(N, \mu(\mathcal{A})) \):

\[ [8a] \quad f(N_\mathcal{A}(\mathcal{A})) = \sum_{\eta \in \mathfrak{P}_k} (-1)^{|\eta|} \binom{|\eta|}{r} f(A_\eta) \]

\[ = \sum_{r \leq \eta \leq p} (-1)^{k-r} \binom{k}{r} S_k, \]

where the \( S_k \) are defined by \([6a]\). Moreover, \([8a]\) satisfies the alternating inequalities.

For \( r = 0 \) we have a formula analogous to \([6c]\) (p. 192).

We use Theorem B (p. 190) once more. For all \( g \in \mathfrak{M}^* \), with supporting atom \( C_0 \) contained in the \( A_i \) such that \( i \in A(C(p)) \), \( I = I_{|\lambda|} \), we have evidently:

\[ [8b] \quad g(N_\mathcal{A}(\mathcal{A})) = 0 \quad \text{if} \quad r \neq I, \quad \text{and} \quad (-1)^I \quad \text{if} \quad r = I. \]

Now the second member of \([8a]\), with \( f \) replaced by \( g \), and \([6d]\) (p. 192) for \((*)\), can be written:

\[ \sum_{\eta \in \mathfrak{P}_k} (-1)^{|\eta|} \binom{|\eta|}{r} g(A_\eta) = \sum_{\eta \in \mathfrak{P}_k} (-1)^{|\eta|} \binom{|\eta|}{r} g(A_\eta) \]

\[ - \sum_{r \leq \eta \leq p} (-1)^{k-r} \binom{k}{r} \binom{l-1}{k} \]

\[ = \binom{l}{r} \sum_{k} (-1)^{k-r} \binom{l-r}{k} \binom{1}{0}. \]
which is indeed equal to [8b]. The alternating inequalities for [8a]
follow from the fact that they hold for \( \sum_k (-1)^{k-r}\left(\begin{array}{c} r-k \\ k \end{array}\right) \), according to
[7e] (p. 194). (The interested reader is referred to [Fréchet, 1940,
1943], as well as to [Takács, 1967], which has a very extensive bibilography.)

We can prove by a similar method:

**Theorem B.** Let \( N_{\geq r}(\mathcal{A}) \) stand for the set of points of \( N \) that are covered
by at least \( r \) subsets of \( \mathcal{A} \), then we have:

\[
\text{[8c]} \quad f(N_{\geq r}(\mathcal{A})) = \sum_{\mathcal{A} \in \mathcal{P}([n])} (-1)^{|\mathcal{A}| - r} \binom{|\mathcal{A}| - 1}{r - 1} f(A_n)
\]

\[
= \sum_{r \leq k < r} (-1)^{k-r} \binom{k-1}{r-1} S_k,
\]

with the alternating inequalities.

4.9. PERMANENTS

**Definition.** Let \( B:=\{b_{ij}\}_{i \leq n, j \leq m} \) be a rectangular matrix with \( m \)
rows and \( n \) columns, \( m \leq n \), with coefficients \( b_{ij} \) in a commutative ring \( \mathcal{O} \).
The permanent of \( B \), denoted by \( \text{per} B \), equals, by definition:

\[
\text{[9a]} \quad \text{per} B = \sum_{\pi \in \mathcal{S}_n} b_{1,\pi(1)} b_{2,\pi(2)} \cdots b_{n,\pi(n)},
\]

where the summation is taken over all \( n \)-arrangements of \( [n] \) (p. 6).

(For the main properties and an extensive bibliography see [Marcus,
Minc, 1965].)

For example, \( \text{per}\begin{pmatrix} 2 & 3 & 1 \\ 5 & 0 & 4 \end{pmatrix} = 2.0 + 5.3 + 2.4 + 5.1 + 3.4 + 0.1 = 40. \)

Hence there are \((n)_m \) terms in the summation [9a]. If \( m=n \), the terms
of \( \text{per}(B) \) are, up to sign, those of \( \det(B) \), and for the permanent there
are properties similar to those of the determinants; however, \( \text{per}(AB) \neq \text{per}(A) \cdot \text{per}(B) \), in general.

For each matrix \( A:=\{a_{ij}\}_{i \leq p, j \leq q} \) with \( a_{ij} \in \mathcal{O} \), let \( w(A) \) be the product
of the \( p \) sums of elements of each row of \( A \):

\[
\text{[9b]} \quad w(A) = \prod_{i=1}^{p} \sum_{j=1}^{q} a_{i,j};
\]

and for every subset \( \lambda \subset [q] \) let \( A(\lambda) \) be the matrix obtained by keeping
in \( A \) precisely those columns whose index belongs to \( \lambda \). For example, if
\( A = \begin{pmatrix} 1 & 3 & 2 & 3 \\ -2 & 4 & 1 & 0 \end{pmatrix} \), then \( w(A) = 9 \times 3 = 27 \) and \( A(\{1, 3\}) = \begin{pmatrix} 1 & 2 \end{pmatrix} \).

**Theorem (Ryser formula, [Ryser, p. 26]).** With the above notations, and
\( w(B(\emptyset))=0 \), \( \text{per}(B) \) is also equal to:

\[
\text{[9c]} \quad \sum_{\lambda \subset \{1, \ldots, q\}} (-1)^{|\lambda|-|\lambda|} \binom{n-|\lambda|}{n-m} w(B(\lambda)),
\]

that is to say

\[
\text{[9d]} \quad \sum_{\lambda \subset \{1, \ldots, q\}} w(B(\lambda)) \binom{n-m+1}{n-m} \sum_{\lambda \subset \{1, \ldots, q\}} w(B(\lambda)) +
\]

\[+ \cdots + (-1)^{|\lambda|-1} \binom{n-1}{n-m} \sum_{\lambda \subset \{1, \ldots, q\}} w(B(\lambda)).
\]

Particularly, for a square matrix, \( m=n \),

\[
\text{[9e]} \quad \text{per} B = \sum_{\lambda \subset \{1, \ldots, q\}} (-1)^{|\lambda|-|\lambda|} w(B(\lambda)) =
\]

\[
= \sum_{\lambda \subset \{1, \ldots, q\}} (-1)^{|\lambda|-1} \sum_{\lambda \subset \{1, \ldots, q\}} w(B(\lambda)).
\]

We use [8a], p. 195. The role of \( N \) is played here by the set of maps
of \( [m] \) into \( [n] \), so \( N=\{[m]\}^{-[n]} \) (caution! \( |N|=n^m \)), with as system
\( \mathcal{A}=(A_1, A_2, \ldots) \)

\[
\text{[9f]} \quad A_i := \{\phi \in [m]^{[n]} \mid \exists j \in [m], \phi(j) = i\}, \quad i \in [n].
\]

Now we suppose first that all \( b_{i,j} \) are real nonnegative. We define the
measure \( f \) for each subset \( X \subset [n] \) to be

\[
\text{[9g]} \quad f(X) := \sum_{\phi \in X} f(\phi), \quad \text{where} \quad f(\phi) := \prod_{i=1}^{m} b_{i,\phi(i)}.
\]

Now \( \phi \) is injective \((\in \mathcal{S}_n)\) if and only if the image of \( [m] \) under \( \phi \) has
cardinality \( m \), in other words, \( \phi \in N_n(\mathcal{A}) \) in the notation of Theorem A
To this expression we will apply now [8a] (p. 195). Let \( x := \{a_1, a_2, \ldots, a_n\} \subseteq [n] \). Then we have:

\[
\varphi(A) := \sum_{\pi \in \mathcal{A}\,\mathcal{A}} b_{\varphi(1)} b_{\varphi(2)} \cdots = \sum_{\pi \in \mathcal{A}\,\mathcal{A}} b_{\varphi(1)} b_{\varphi(2)} \cdots,
\]

where \( \varphi \) stands for the set of maps of \([n]\) into \([n] - x \). Hence, by Theorem A (p. 127), and the notation of [9b]:

\[
\Phi\left(\mathcal{A}\,\mathcal{A}\right) = \omega\left(\mathcal{A}\,\mathcal{A}\right).
\]

Then [9c] follows by putting \( \lambda := [n] - x \) in [9i] and [8a] (p. 195). Since [9c] is true for all \( b_{ij} \geq 0 \), it is also true in a commutative ring, since the term-by-term expansion [9a] is the same in both cases. ■ (For other expressions of \( \text{per} \), see [*Cartier, Foata, 1969*], p. 76, [Crapo, 1968], [Wilf, 1968a, b]).

If \( \text{per} \) can be directly computed, then [9a] gives, together with [9c, d, e], a 'remarkable' identity. For example, when \( B \) is the square matrix of order \( n \) consisting entirely of 1, then clearly \( \text{per} B = n^n \). Hence by [9e]:

\[
\sum_{i \leq n} (-1)^{n-i} \binom{n}{i} \mu(x) = n! \sum_{i \leq n} (-1)^{n-i} \binom{n}{i} \mu(x).
\]

Thus we find back the evident property \( S(n, n) = 1 \) for the Stirling numbers ([1b] p. 204). If we take next \( b_{ij} = 2^{j-i} \), we find [9j]

\[
\sum_{i \leq n} (-1)^{n-i} \binom{n}{i} \mu(x) = \sum_{i \leq n} (-1)^{n-i} \binom{n}{i} \mu(x).
\]

Finally, if all \( b_{ij} = 0 \), except \( b_{1,1} = 1 = b_{2,2} = \cdots = b_{n,n,x} = 1 \) and \( b_{1,2} = b_{2,3} = \cdots = b_{n-1,n} = 1 \), we find, using [9b] (p. 24):

\[
x^y + y^x = \sum_{k \leq n/2} (-1)^k \frac{n-k}{n-k} \binom{n-k}{k} (xy)^k (x+y)^{n-k},
\]

to be compared with Exercise 1, p. 155.

**SUPPLEMENT AND EXERCISES**

1. **Variegated words.** Using 2 letters \( a_1 \), 2 letters \( a_2 \), ..., 2 letters \( a_n \), how many words of length \( 2n \) can be formed in which no two identical letters

are adjacent? (For instance, for \( n = 3 \), the word \( a_3a_2a_1a_2a_3 \) [Hint: When \( A_i \) stands for the set of words in which the two letters \( a_i \) are adjacent, then the required number is equal to \( |A_1A_2 \ldots A_n| \).] Now generalize. (Cf. Exercise 1, p. 219, and Exercise 21 (3), p. 265.)

2. **Sums of the type of the Euler function.** If in the following the summation is taken over all integers \( x < n \) which are prime relatively to \( n, n = p_1p_2 \cdots \cdots p_r \), then show that \( \sum x^2 = (n^2/3) \varphi(n) + (-1)^{r} \prod_{j=1}^{r} p_j \varphi(n) \). Generalize to \( \sum x^n \).

3. **Jordan function.** This is the following double sequence:

\[
J_k(n) := n^k \prod_{p|n} (1 - p^{-k}),
\]

\( p \) is a prime number, and where \( p | n \) means '\( p \) divides \( n \). It is a generalization of the Euler function ([6e] p. 193) \( J_1(n) = \varphi(n) \). For any integer \( k \geq 1 \), show that \( J_{k}(n) \) is equal to the number of \((k+1)\)-tuples \( (x_1, x_2, \ldots, x_n) \) of integers \( x_i \in [n], i \in [k] \), whose GCD equals \( 1 \). Show that \( \sum_{d|n} J_k(d) = n^k \) and derive from this the Lambert GF (Exercise 16, p. 161) \( \sum_{n \geq 1} J_k(n) t^n = (1 - t)^{k-1} A_k(t) \) \((1 - t)^{-k-1} \) where the \( A_k(t) \) are the Eulerian polynomials of p. 244.

4. **Other properties of the number \( d(n) \) of derangements.** (1) We have \( d(n) = n! A^n \), \( A \) being the difference operator (p. 13). (2) \( f = \sum d(n) t^n \) satisfies the differential equation \((t^2 + t) f + (t^2 - 1) f + 1 = 0 \). Use this to prove: \( f = -t^{-1} \exp(-t^{-1}) \) \( \exp(t^{-1}) \) \((t + t^2)^{-1} \) \( \exp(-t(1-u)) \) formally. (3) The number \( d_n(n) \) of permutations of \([n]\) with \( k \) fixed points (GF, p. 231) has as: \( \sum_{k \geq 0} d_n(n) t^k n! = (1 - t)^{-1} \exp(-t(1-u)) \).

5. **Other properties of the reduced ménages numbers \( \mu(n) \).** (1) The following recurrence relation holds: \( (n-2) \mu(n) - (n-2) \mu(n-2) + n \mu(n-2) = 4 (-1)^{n+1} \) ([*Lucas, 1891*], p. 495). (2) When \( n \) tends to infinity, \( \mu(n) \sim n! e^{-2} \). (3) \( n! = \sum_{k=0}^{n} \binom{2n}{k} \mu(n-k), \mu(0) = 1, \mu(1) = 1 \) (Riordan). (4) \( \mu(n) = \#e^{-2} \sum (-1)^k (n-k-1)! k!, \) where \( 0 \leq k \leq \leq (n-1)/2 \), with the notation [6f] (p. 110) (Schöbe). (5) \( \sum_{n \geq 3} \mu(n) t^n = \)
= (t^2 - 1) t^{-4} \exp(-t - t^{-1}) \int t^2 (t+1)^{-2} \exp(t + t^{-1}) dt, \ldots \text{ formally (Cayley, 1878b)}. 

6. Random integers. Repetitions being allowed, \( n \) integers \( \geq 1 \) are independently drawn at random, say \( \omega_1, \omega_2, \ldots, \omega_n \). What is the probability that the product \( \pi_n := \omega_1 \omega_2 \ldots \omega_n \) has last digit (the number of units, hence) equal to \( 5 \)? More generally, compute the probability that a given integer \( k \geq 1 \) divides \( \pi_n \).

7. Knock-out tournaments. A set of \( 2^t \) players of equal strength is at random arranged into \( 2^{t-1} \) disjoint pairs. They play one round, and \( 2^{t-1} \) are eliminated. The same operation is repeated with the remaining \( 2^{t-1} \) players, until a champion remains after the \( t \)-th round. Show that the probability that a player takes part in exactly \( i \) rounds equals \( \frac{2^{-i}}{i} \) for \( 1 \leq i \leq t-1 \) and \( 2^{-t} + \frac{1}{i} \) if \( i = t \). (Narayana, 1968, and Narayana, Zidek, 1968) for other results and generalizations. See also [André, 1900].

8. A determinant. Let \( A \) be a square matrix of order \( n \), \( A := [a_{i,j}]_{i,j \in [n]} \), where the \( a_{i,j} \) belong to a commutative ring \( \Omega \). For each subset \( \kappa \subseteq [n] \), let \( D(\kappa) \) be the determinant of the matrix that is obtained by deleting from \( A \) all rows and columns whose index does not belong to \( \kappa \), \( D(\emptyset) := 1 \). Then, for \( x_1, x_2, \ldots \in \Omega \):

\[
\begin{vmatrix}
 a_{1,1} + x_1 & a_{1,2} & \cdots & a_{1,n} \\
 a_{2,1} & a_{2,2} + x_2 & \cdots & a_{2,n} \\
 \vdots & \vdots & \ddots & \vdots \\
 a_{n,1} & a_{n,2} & \cdots & a_{n,n} + x_n
\end{vmatrix}
= \sum_{\kappa \subseteq [n]} \{ D(\kappa) \prod_{i \in \kappa} x_i \}.
\]

9. Inversion of the Jordan formula. In [8a] (p. 195) we put \( T_* := f(N_q(\alpha)) \), \( T_* = \sum_k (-1)^{k-r} \binom{k}{r} S_k \). Now show that \( S_k = \sum \binom{k}{r} T_k \).

10. Inequalities satisfied by the \( S_k \). Show that the \( S_k \), as defined by [6a] (p. 192), satisfy the Fréchet inequalities ([Fréchet, 1940]):

\[
S_k \bigg/ \binom{p}{k} \leq S_{k-1} \bigg/ \binom{p}{k-1}, \quad k \in \mathbb{Z}.
\]

and the Gumbel inequalities, \( k \in \mathbb{Z} \):

\[
\left\{ \binom{p}{k+1} S_{k+1} \right\} \bigg/ \left\{ \binom{p}{k} S_k \right\} \leq \left\{ \binom{p}{p} S_{k+1} \right\} \bigg/ \left\{ \binom{p}{k-1} S_k \right\}.
\]

11. The number of systems of distinct representatives. Let \( \mathcal{B} := (B_1, B_2, \ldots, B_m) \) be a system of not necessarily distinct blocks of \( [n] \), \( B_i \subseteq [n] := \{1, 2, \ldots, n\} \), \( 1 \leq m \leq n \), and let \( \mathbf{B} = [b_{i,j}] \) be the incidence matrix of \( \mathcal{B} \) defined by \( b_{i,j} = 1 \) if \( j \in B_i \) and \( = 0 \) otherwise, \( i \in [m], j \in [n] \). Show that the number of systems of distinct representatives (Exercise 32, p. 300) of \( \mathcal{B} \) equals \( \text{per} (\mathbf{B}) \).

12. Permanent of a stochastic matrix. Let \( A := [a_{i,j}] \) be a \( n \times n \) square double stochastic matrix. This means:

\[
a_{i,j} \geq 0, \quad \sum_{j=1}^{n} a_{i,j} = 1, \quad \sum_{i=1}^{n} a_{i,j} = 1, \quad i, j \in [n].
\]

Let \( n \) boxes contain each a ball. At a certain moment, each ball jumps out of its box, and falls back into a random box (perhaps the same) such that the ball from box \( i \) goes to box \( j \) with a probability of \( a_{i,j} \). Then, \( \text{per} A \) represents the probability that after the transfer there still is one ball in each box.

13. The number of permutations with forbidden positions. Let \( I \) stand for the \( n \times n \) unit matrix, and let \( J \) be the \( n \times n \) matrix, all whose entries equal 1. Then show that \( \text{per} (J-I) = d(n) \), the number of derangements of \( [n] \) (p. 180). Use this to obtain (by [9c] p. 197):

\[
d(n) = \sum_{r=0}^{n-1} (-1)^r \binom{n}{r} (n-r) (n-r-1)^r.
\]

More generally, let \( \mathcal{B} \) be a relation in \( [n] \), \( \mathcal{B} \subseteq [n] \times [n] \), and let \( \mathcal{G}_m(n) \) be the set of permutations \( \sigma \) of \( [n] \) such that \( i, \sigma(i) \in \mathcal{B} \). Let also \( \mathcal{B} = [b_{i,j}] \) be the \( n \times n \) square matrix such that \( b_{i,j} = 1 \) for \( (i, j) \in \mathcal{B} \), and \( = 0 \) otherwise. Then \( \mathcal{G}_m(n) = \text{per} (\mathbf{B}) \). (There is in [Riordan, 1958], pp. 163–237, a very complete treatise on this subject. See also [Foata, Schützenberger, 1970].)

14. Vector spaces. Let \( A_1, A_2, \ldots \) be finite dimensional vector subspaces
with dimensions $\delta(A_1), \delta(A_2), \ldots$. We denote $A_1A_2$ for $A_1 \cap A_2$. Then (1) $\delta(A_1 + A_2) = \delta(A_1) + \delta(A_2) - \delta(A_1A_2)$, where $A_1 + A_2$ stands for the subspace spanned by $A_1 \cup A_2$. (2) $\delta(A_1 + A_2 + A_3) = \delta(A_1) + \delta(A_2) + \delta(A_3) - \delta(A_1A_2) - \delta(A_1A_3) - \delta(A_2A_3) + \delta(A_1A_2A_3)$. (3) This inequality cannot be generalized to more than three subspaces; but we always have: $\delta(A_1A_2 \ldots A_n) \leq \delta(\sum_{i=1}^{n} A_i) = \sum_{i=1}^{n} \delta(A_i)$.

*15. Möbius function. Let $P$ be a partially ordered set, in other words, there is an order relation $\leq$ given on $P$ (Definition D, p. 59). Moreover, $P$ is supposed to be locally finite, in the sense that each segment $[x, y] := \{t \in P \mid x \leq t \leq y\}$ is finite. $A$ stands for the set of functions $f: x, y \in P$, real-valued, which are zero if $x \not\leq y$ (i.e., not $x \leq y$). (1) We define the (convolution) product $h$ of $f$ by $g$, denoted by $h = f * g$, by:

$$h(x, y) := \sum_{x \leq y} f(x, u) g(u, y).$$

Show that with this multiplication, $A$ becomes a group, with unit element $6$ defined by $6(x, y) = 1$ for $x = y$, and $6 = 0$ otherwise. (2) The zeta function $\zeta$ of $P$ is such that $\zeta(x, y) = 1$ if $x \leq y$ and $\zeta = 0$ otherwise. The inverse $\mu$ of $\zeta$, which satisfies $\mu \cdot \zeta = 6 \cdot \mu = 6$, is called the Möbius function of $P$. If we suppose that $P$ has a universal lower bound denoted by $0$, verify the following 'Möbius inversion formula' for $f, g \in A$:

$$\mu(x, y) = \sum_{x \leq y} f(y) \leftrightarrow f(x) = \sum_{x \leq y} g(y) \mu(y, x).$$

(3) Let $P := \{1, 2, 3, \ldots\}$ be ordered by divisibility: $x \leq y \iff x | y \implies x | y$ divides $y$. Show that $\mu(x) = 1$; $\mu(x, y) = (-1)^k$ if $x$ divides $y$ and the quotient equals $p_1 p_2 \ldots p_k$ where the prime numbers $p_i$ are all different; $\mu(x, y) = 0$ in the other case. Hence $\mu(x, y) = \bar{\mu}(y | x)$, where $\bar{\mu}(n)$ is the ordinary arithmetical Möbius function (Exercise 16, p. 161). What does the inversion formula (3) give us in this case? (4) We order the set $P := \mathcal{P}(N)$ of subsets of a finite set $N$ by inclusion. Then $\mu(x, y) = (-1)^{[y]-[x]}$ if $x \leq y \iff (x \subseteq y)$. What does (3) give in this case? (5) Let $P$ now stand for the set of partitions of a finite set $N$ ordered as in Exercise 3 (p. 220). Then, for $x \leq y$ with $y = \{B_1, B_2, \ldots, B_k\}$, $B_1 + B_2 + \ldots + B_k = N$, we have $\mu(x, y) = (-1)^{[y]-[x]}(n_1) \ldots (n_k - 1)$, where $n_i$ is the number of blocks of $x$ contained in $B_i$, $\leq [k]$. (This formula is due to Schützenberger, 1954). For a recent study of all these questions see [Rota, 1964b] and [*Cartier, Foata, 1969], pp. 18-23. See also [Weisner, 1935], [Frucht, Rota, 1963], [Crapo, 1966, 1968], [Smith, 1967, 1969].

*16. Jordan and Bonferroni formulas in more variables. Let $A_1, A_2, \ldots, A_p$ and $B_1, B_2, \ldots, B_q$ be subsets of $N$, and let $N_{r,s}$ be the set of points of $N$ belonging to $r$ sets $A_i$ and to $s$ sets $B_j$. For each measure $f$ on $N$, we put $S_{r,s} := \cup f(A_i B_j)$, where $g \in \mathcal{P}(\mathcal{P}(N))$ and $\lambda \in \mathcal{P}(\mathcal{P}(N))$, with notation [1e] of p. 177.

(1) $f(N_{r,s}) = \sum_{i=1}^{r+q} \sum_{i+j=r} (-1)^{(i+j)} \left( \begin{array}{c} i \\begin{array}{c} j \end{array} \end{array} \right) S_{i, j}.$

(2) With a notation analogous to that of Theorem B (p. 196):

$$f(N_{r,s}) = \sum_{i=1}^{r+q} \sum_{i+j=r} (-1)^{(i+j)} \left( \begin{array}{c} i-1 \\begin{array}{c} j-1 \end{array} \end{array} \right) S_{i, j}.$$

(3) With respect to the first summations in (1) and (2) the alternating inequalities hold ([Meyer, 1969]).

(4) Generalize to more than two systems of subsets of $N$.

*17. A beautiful determinant. Let $(i,j)$ be the GCD of the integers $i$ and $j$, and let $\varphi(k)$ be the Euler function (p. 193). Show that:

$$\left( \begin{array}{cccc} 1, 1 & 1, 2 & \cdots & 1, n \\ 2, 1 & 2, 2 & \cdots & 2, n \end{array} \right) = \varphi(1) \varphi(2) \cdots \varphi(n)$$

$$\cdots \cdots \cdots$$

$$\left( \begin{array}{cccc} n, 1 & n, 2 & \cdots & n, n \end{array} \right).$$

([Smith, 1875], [Catalan, 1878]).

More generally, if we replace in the preceding every $(i,j)$ by $(i,j)'$, then the determinant equals $\prod_{i=1}^{n} J_s(k)$, where $J_s(k)$ is the Jordan function of Exercise 3 (p. 199).
STIRLING NUMBERS

Let us give a survey of the three most frequently occurring notations:
numbers of the first kind $s(n, k)$ (Riordan, and also this book, ...) $= S_n^k$ (Jordan, Mitrinović, ...)$= (-1)^{n-k}S_1(n-1, n-k)$ (Gould, Hagen, ...);
numbers of the second kind $S(n, k) = Q_n^n = S_2(k, n-k)$.

5.1. STIRLING NUMBERS OF THE SECOND KIND $S(n, k)$ AND PARTITIONS OF SETS

DEFINITION A. The number $S(n, k)$ of $k$-partitions (partitions in $k$ blocks, Definition C, p. 30) is called Stirling number of the second kind. Hence $S(n, k) > 0$ for $1 < k \leq n$ and $S(n, k) = 0$ if $1 < n < k$.

We put $S(0, 0) = 1$ and $S(0, k) = 0$ for $k \geq 1$.

In other words, $S(n, k)$ is the number of equivalence relations with $k$ classes on $N$. It is also the number of distributions of $n$ distinct balls into $k$ indistinguishable boxes (the order of the boxes does not count) such that no box is empty.

On p. 206 we will prove that the $S(n, k)$ are indeed the number previously introduced on p. 50.

THEOREM A. The Stirling number of the second kind $S(n, k)$ equals:

$$S(n, k) = \frac{1}{k!} \sum_{0 \leq j \leq k} (-1)^j \binom{k}{j} (k-j)^n =$$

$$= \frac{-1}{k!} \sum_{1 \leq i \leq k} (-1)^{k-i} \binom{k}{i} \binom{n}{i} \binom{\binom{n}{i}}{i}$$

[le] and the formula is still true for $k > n$ ($\Rightarrow S(n, k) = 0$, [1a]).

For the proof of [1b, (●)] we apply the sieve method of p. 177. Let $E$ be the set of maps of $N$ into $[k] := \{1, 2, \ldots, k\}$ and let $F$ be the subset of $E$ consisting of the surjective maps:

$$[1d] \quad |E| = k^n, \quad |F| = k! S(n, k),$$

(●) follows from [3a] (p. 4) and (●) from the fact that any $f \in E$ corresponds to precisely one partition of $N$, namely the partition consisting of the $k$ pre-images $f^{-1}(i)$, $i \in [k]$ (p. 30), together with a numbering of this partition. Let now $B_i$ be the set of $f \in E$ that do not have $i$ in their image: $\forall x \in N, f(x) \neq i$. Evidently $E = B_1 \cup B_2 \ldots \cup B_k$ and for the interchangeable system of the $B_i$ (p. 179), we have $|B_1| = |B_2| = \ldots = |B_k| = \binom{n}{j+1} = \binom{j+2}{k}$. Hence, by [1m] (p. 180), for (§):

$$[1e] \quad k! S(n, k) = |F| = \sum_{j=1}^{k} \binom{k}{j} |B_j| = \sum_{j=1}^{k} \binom{k}{j} |B_j| + \sum_{j=1}^{k} \binom{k}{j} |B_j| = \ldots = CQFD.$$

As far as [1b(●●)] is concerned, this is formula [6f] (p. 14). Finally, if $n < k$, then $|F|$ is clearly equal to 0 and the sieve formula can still be applied, hence [1c].

Thus we find $S(n, 1) = 1$, $S(n, 2) = 2^{n-1} - 1$, $S(n, 3) = (3^{n-1} + 1)/2 - 2^{n-1} \ldots$. Another way to prove [1b] would be to observe that any map $f \in E$ is surjective from $N$ onto $I := f(N)$. So, putting $u_k := k! S(n, k)$,

$$u_k := |E| = k^n = \sum_{i=1}^{k} u_i i! = \sum_{i=1}^{k} \binom{k}{i} u_i$$

[le] which gives $u_k$ (consequently $S(n, k)$) by the inversion formula [6e] p. 144.

DEFINITION B. A partition $\mathcal{S}$ of a set $N$ is said to be of type $[c] = [c_1, c_2, \ldots, c_n]$, where the integers $c_i \geq 0$ satisfy $c_1 + c_2 + \ldots + c_n = n (= |N|)$, if and only if $\mathcal{S}$ has $c_1$ 1-blocks, $c_2$ 2-blocks, ..., $c_n$ n-blocks, $c_1 + c_2 + \ldots + c_n = |\mathcal{S}|$.

THEOREM B. The number of partitions of type $[c]$ is equal to $n!/(c_1! c_2! \cdots (1!)^{c_1} (2!)^{c_2} \cdots)$.

Given such a partition is equivalent to first giving a division of $N$ into $c_1$ 1-blocks, $c_2$ 2-blocks, ..., of these there are $z = n!/(1!)^{c_1} (2!)^{c_2} \cdots$, [10c] (p. 27); and to consequently erasing the numbering of blocks with equal size; so we must divide the number $z$ by $c_1! c_2! \cdots$. ■
5.2. Generating functions for $S(n, k)$

The following theorem shows that the Stirling numbers defined in [1b] are indeed the numbers which were introduced for the first time in [14s] (p. 51).

**Theorem A.** The Stirling numbers of the second kind $S(n, k)$, have as 'vertical' GF:

\[
\Phi_k(t) := \sum_{n \geq k} S(n, k) \frac{t^n}{n!} = \frac{1}{k!} (e^t - 1)^k, \quad k \geq 0
\]

where $n \geq k$ can be replaced by $n \geq 0$, and for 'double' GF:

\[
\Phi(t, u) := \sum_{n, k \geq 0} S(n, k) \frac{t^n u^k}{n!} = 1 + \sum_{n \geq 1} \left\{ \sum_{1 \leq j \leq n} S(n, k) u^k \right\} = \exp \{ u (e^t - 1) \}.
\]

**Theorem B.** The $S(n, k)$ have for 'horizontal' GF (which is often taken as definition of the $S(n, k)$):

\[
\varphi_k := \sum_{n \geq k} S(n, k) u^n = (1 + (e^u - 1))^k, \quad k \geq 1.
\]

Identify the coefficients of $t^n/n!$ in the first and last member of:

\[
\sum_{n \geq 0} x^n \frac{t^n}{n!} = (1 + (e^t - 1))^x = \sum_{k \geq 0} (x)_k \frac{(e^t - 1)^k}{k!} = \sum_{0 \leq i \leq n} (x)_k S(n, k) \frac{t^n}{n!}.
\]

where $(*)$ follows from [12e] (p. 37), and $(**)$ from [2a].

**Theorem C.** The $S(n, k)$ have the following rational GF:

\[
\varphi_k := \sum_{n \geq k} S(n, k) u^n = \frac{1}{(1 - u)(1 - 2u) \cdots (1 - ku)}, \quad k \geq 1.
\]

(According to [1a], $n \geq k$ can be replaced by $n \geq 0$.)

If we decompose the rational fraction $\varphi_k$ into partial fractions, we obtain equality $(*)$, and for $(**)$ we use [1b]. Then we get:

\[
\varphi_k = \frac{u^k}{(1 - u)(1 - 2u) \cdots (1 - ku)} = \sum_{0 \leq i \leq k} \left\{ \frac{(-1)^i}{k!} \binom{k}{j} \sum_{n \geq 0} (k - j)^n u^n \right\}.
\]

Similarly, with [2a] for $(**)$:

\[
\Phi(t, u) = \sum_{n \geq 0} \left\{ u^k \sum_{j \leq k} \frac{(e^j - 1)^j}{n!} \right\} = \sum_{n \geq 0} \frac{1}{k!} \binom{k}{j} \sum_{0 \leq i \leq k} (k - j)^n u^n.
\]

**Theorem D.** The following explicit formula holds:

\[
S(n, k) = \sum_{\xi_1 + \xi_2 + \cdots + \xi_n = n - k} 1^{\xi_1} 2^{\xi_2} \cdots k^{\xi_n}.
\]
In other words, the Stirling number of the second kind $S(n, k)$ is the sum of all products of $n-k$ not necessarily distinct integers from $[k] = \{1, 2, \ldots, k\}$ (there are $\binom{n-1}{k-1}$ such products).

For instance, $S(5, 2) = 1^2 + 2^2 + 1^2 + 2^2 + 2^2 = 15$. Thus the numbers $S(n, k)$ are the symmetric monomial functions of degree $(n-k)$ of the first $k$ integers (Exercise 9, p. 158). This is the same thing as expanding $(1 + 2 + \cdots + k)^{n-k}$ by the multinomial theorem and afterwards suppressing every multinomial coefficient. (This procedure applied to $(a_1 + a_2 + \cdots + a_k)^n$ gives the so-called Wronski alephs.)

After expanding $\varphi_k$, \[2d\], identify the coefficients of $u^{n-k}$ of the first and last member of:

$$
\varphi_k u^n = \prod_{j \leq k} (1 - j u)^{-1} = \prod_{j \leq k} \sum_{\xi \geq 0} j^\xi u^j = \sum_{\xi_1, \xi_2, \ldots, \xi_k \geq 0} (\xi^1 2^{\xi_2} \cdots k^{\xi_k}) u^{1+2+\cdots+k}. 
$$

5.3. Recurrence relations between the $S(n, k)$

**Theorem A.** The Stirling numbers of the second kind $S(n, k)$ satisfy the 'triangular' recurrence relation:

\[3a\]  
$$S(n, k) = S(n-1, k-1) + k S(n-1, k), \quad n, k > 1; 
S(n, 0) = S(0, k) = 0, \quad \text{except} \quad S(0, 0) = 1. 
$$

This is a quick tool for computing the first values of $S(n, k)$ (see table on p. 310).

We give two proofs of \[3a\].

1) **Analytical.** Equate the coefficients of $(x)_{k}$ in the first and last members of $\[3b\]$:  

\[3b\]  
$$  
\sum_{k} S(n, k) (x)_{k} = x^{n} = x \cdot x^{n-1} = x \cdot \sum_{h} S(n-1, h) (x)_{h}  
= \sum_{h} S(n-1, h) \{ (x)_{h+1} + h (x)_{h} \},  
$$

since the $(x)_{h}$ form an independent system of vectors in the linear space of polynomial functions.

(2) **Combinatorial.** We return to Definition A (p. 204) of the $S(n, k)$. Let $x \in \mathbb{N}$ be a fixed point, and let $M := N - \{x\}$, $|M| = n \geq 2$. We partition the set $s = s(N, k)$ of the $k$-partitions of $N$ into $s'$ and $s''$, $s'$, $s''$, is the set of partitions in which the block $\{x\}$ occurs, and $s'' = s - s'$. For all $\mathcal{F} \in s'$, let $\tau(s') = (B \cap M) 1_{B \in s'}, B \cap M = \emptyset$ be the trace of $s'$ on $M$. If $\mathcal{F} \in s''$, $\tau(s') \in s(M, k-1)$, and we see clearly that $\tau$ is bijective; hence $|s''| = |s(M, k-1)| = S(n-1, k-1)$. If $\mathcal{F} \in s'$, $\tau(s') \in s(M, k)$, and for each partition $\mathcal{F} \in s(M, k)$, $|\tau^{-1}(\mathcal{F})|$ equals the number of possible choices of joining $x$ to one of the blocks of $\mathcal{F}$, which is $k$; hence $|s''| = k |s(M, k)| = k S(n-1, k)$. Finally, \[3a\] follows from $|s| = |s'| + |s''|$. ■

**Theorem B.** The $S(n, k)$ satisfy the 'vertical' recurrence relations:

\[3c\]  
$$S(n, k) = \sum_{k \leq l \leq n-1} \binom{n-1}{l} S(l, k-1). 
$$

\[3d\]  
$$S(n, k) = \sum_{k \in \mathbb{N}} S(l-1, k-1) k^{l-1}. 
$$

For \[3c\], we differentiate \[2a\] (p. 206) with respect to $t$, and we identify the coefficients of $t^{-1} (t^{n-1})$, in the first and last member of:

$$
\sum_{n \geq 0} S(n, k) \frac{t^{-1}}{(n-1)!} = \frac{d \varphi_k}{dt} = e^{t} \Phi_{k-1} = \sum_{l, m \geq 0} S(l, k-1) \frac{t^{l+m}}{l! m!}. 
$$

For \[3d\], use \[2d\] (p. 207):

$$
\sum_{n \geq k} S(n, k) u^{n} = \varphi_k = n (1 - ku)^{-1} \varphi_{k-1} = \sum_{l, m \geq 0} S(l, k-1) k^{l+m}. 
$$

**Theorem C.** The $S(n, k)$ satisfy the 'horizontal' recurrence relations:

\[3e\]  
$$S(n, k) = \sum_{0 \leq j \leq n-k} (-1)^j \langle k+1 \rangle \cdot S(n+1, k+j+1)  
$$

where  
$$\langle k \rangle := k(k+1) \cdots (k+j-1), \quad \langle k \rangle := 1. 
$$

\[3f\]  
$$k! S(n, k) = k^n - \sum_{j=1}^{k-1} \langle k \rangle S(n, j). 
$$

It suffices, by \[3a\], to replace $S(n+1, k+j+1)$ of \[3e\] by $S(n, k+j) +$
The number \( w(n) \) of all partitions of a set \( N \), often called exponential number or Bell number ([Becker, Riordan, 1948], [Touchard, 1956]) apparently equals, by Definition A (p. 204):

\[
[4a] \quad w(n) = \sum_{1 \leq k \leq n} S(n, k), \quad n \geq 1.
\]

So it is also equal to the number of equivalence relations on \( N \).

**Theorem A.** The numbers \( w(n) \) have the following GF:

\[
[4b] \quad \sum_{n \geq 0} w(n) \frac{t^n}{n!} = \exp(e^t - 1), \quad w(0) := 1.
\]

They satisfy the recurrence relations ([Aitken, 1933]):

\[
[4c] \quad w(n+1) = \sum_{0 \leq k \leq n} \binom{n}{k} w(k), \quad n \geq 0,
\]

and they can be given in the form of a convergent series ([Dobinski, 1877]):

\[
[4d] \quad w(n) = \frac{1}{e} \sum_{k=0}^{n} \frac{H^n_k}{k!} = \frac{1}{e} \sum_{k=0}^{n} \frac{H^n_k}{k!} (n \geq 1, [6f] p. 110).
\]

Taking into account \([4a]\), the first member of \([4b]\) equals \( \Phi(t, 1) \), then, by \([2b]\) (p. 206) the result follows.

For \([4c]\), as for \([3a]\), there are two ways again. *Analytically,* identify the coefficients of \( t^n/n! \) in \( \Phi(t, 1) \). *Combinatorially,* let \( s(P) \) be the set of all partitions of \( P \), \( |P| = n+1 \) and let \( x \in P \) be a fixed point, \( N := P - \{x\}, |N| = n \). For \( K \subseteq N \), let \( s_K(P) \) be the set of partitions of \( P \) such that the block containing \( x \) is \( \{x\} \cup K \). Then we have evidently a bijection between \( s(N - K) \) and \( s_K(P) \). Hence, by virtue of the division \( s(P) = \sum_{K \subseteq N} s_K(P) \) and by passage to the cardinals, we have:

\[
\begin{align*}
\sum_{n \geq 0} w(n) \frac{t^n}{n!} &= \exp(e^t - 1) = \sum_{k=0}^{\infty} \frac{t^k}{k!} = \sum_{k=0}^{\infty} \frac{(1 - t)^{-k}}{k!} \quad (n \geq 1, [6f] p. 110) \quad (\star)
\end{align*}
\]

We are leaving \([4d]\) (*) to the reader as a gift. ■

See [Rota, 1964a] and its bibliography. (For the asymptotic study of \( w(n) \) see [Moser, Wyman, 1955b], [Binet, Szekeres, 1957], and [*De Bruijn, 1961], pp. 102–8. See also Exercise 23, p. 296.) A table of \( w(n) \) is found on p. 310.

We show now a method of computation of the \( w(n) \) without using the \( S(n, k) \).

**Theorem B.** ([Aitken, 1933]). In the sense of p. 14 we have:

\[
w(n) = A^n w(1).
\]

In fact, by \([6c]\) (p. 13) (here, \( x = 1 \)) for \((\star)\), and by \([4c]\) (p. 210) for \((\star\star)\) we have:

\[
A^n w(1) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} w(k + 1) = \sum_{j=0}^{n} \binom{n}{j} w(j) \binom{n-j}{j} = \sum_{j=0}^{n} w(j) \binom{n-j}{j} (A(n, j),
\]

where

\[
A(n, j) = \sum_{k} (-1)^{n-k} \binom{n}{k} \binom{k}{j} = C_n (1 - j)^n t^n (1 - t)^{-j-1} = C_n (1 - j)^n t^n (1 - t)^{-j-1} = 0, \quad \text{except } A(n, n) = 1, \quad \text{QED.} \quad (\star\star)
\]

More generally, the same method enables to prove that the polynomials \( S_n(x) := \sum_{k=1}^{n} S(n, k) x^k \) satisfy \( x S_n(x) = A^n S_1(x) \) (\( w(n) = S_n(1) \)). In practice, the computation of the \( w(n) \) by way of this property proceeds as in the table shown. One goes from left to right, upward under an...
angle of 45°, starting from the table obtained for \(\sigma(n-1)\). Then, after having arrived at the value of \(\sigma(n)\), it is brought down to the bottom of the first column, and one starts again. In the table is shown the computation of \(\psi(6)\), starting from the table obtained by computing \(\psi(5)\): 52 + 15 = 67, 67 + 20 = 87, etc.

<table>
<thead>
<tr>
<th>(n)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\psi(n))</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>15</td>
<td>52</td>
<td>203</td>
<td>877</td>
</tr>
<tr>
<td>(\delta \psi(n))</td>
<td>1</td>
<td>3</td>
<td>10</td>
<td>37</td>
<td>51</td>
<td>674</td>
<td></td>
</tr>
<tr>
<td>(\delta^2 \psi(n))</td>
<td>2</td>
<td>7</td>
<td>27</td>
<td>114</td>
<td>523</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\delta^3 \psi(n))</td>
<td>5</td>
<td>20</td>
<td>87</td>
<td>409</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\delta^4 \psi(n))</td>
<td>15</td>
<td>67</td>
<td>322</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\delta^5 \psi(n))</td>
<td>52</td>
<td>255</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5.5. Stirling numbers of the first kind \(s(n, k)\) and their generating functions

We have already met two definitions of the Stirling numbers of the first kind \(s(n, k)\):

1. The \(s(n, k)\) have for 'double' GF ([14p], p. 50):

\[
\Psi(t, u) = \sum_{n, k \geq 0} s(n, k) \frac{t^n u^k}{n!} = 1 + \sum_{n=1}^\infty \frac{t^n}{n!} \left( \sum_{k=1}^{\min(n, k)} s(n, k) u^k \right) = (1 + t)^u.
\]

or for 'vertical' GF ([14r], p. 51):

\[
\Psi_k(t) = \sum_{n \geq k} s(n, k) \frac{t^n}{n!} = \frac{1}{k!} \log^k(1 + t);
\]

hence \(s(n, k) = 0\) if not \(1 \leq k \leq n\) except \(s(0, 0) = 1\).

2. The infinite (lower) triangular matrix of the \(s(n, k)\) is the inverse of the matrix of the \(S(n, k)\), [6f] (p. 144):

\[
[s(n, k)] = [S(n, k)]^{-1}.
\]

The \(s(n, k)\) are not all positive, their sign is given by:

\[
|s(n, k)| = (-1)^{k+n} s(n, k)
\]

On p. 235 we will give the combinatorial interpretation of \(|s(n, k)|\), the unsigned or absolute Stirling number of the first kind, which may be denoted by \(s(n, k)\):

\[
s(n, k) := |s(n, k)| = (-1)^{k+n} s(n, k).
\]

Theorem A. The \(s(n, k)\) have for 'horizontal' GF (this is often taken as definition of the \(s(n, k)\)):

\[
\begin{align*}
\Psi_0(u) &= \sum_{n=0}^{\infty} s(n, k) u^n = (1 - u)(1 - 2u) \ldots (1 - (n - 1)u), \\
\Psi_n(u) &= \sum_{n=0}^{\infty} s(n, k) u^n = (1 + u)(1 + 2u) \ldots (1 + (n - 1)u).
\end{align*}
\]

\[\text{THEOREM A.}\] It suffices, by [12e, e'] (p. 37) to identify the coefficients of \(t^n/n!\) in:

\[
\begin{align*}
\sum_{n, k \geq 0} s(n, k) \frac{t^n}{n!} x^k &= \sum_{n=0}^{\infty} (x)^n_n \frac{t^n}{n!}, \\
\sum_{n=0}^{\infty} s(n, k) \frac{t^n}{n!} x^k &= \sum_{n=0}^{\infty} \langle x \rangle^n_n \frac{t^n}{n!}.
\end{align*}
\]

Theorem B. The \(s(n, k)\) have for 'horizontal' GF:

\[
\Psi_n(u) = \sum_{k=0}^{\infty} s(n, k) u^{n-k} = (1 - u)(1 - 2u) \ldots (1 - (n - 1)u)
\]

\[\text{THEOREM B.}\] Replace \(x\) by \(u^{-1}\) in [5e, f'], and simplify. 

Theorem C. The \(s(n+1, k+1)\), for \(n\) fixed and variable \(k\), are the elemen-
tary symmetric functions of the first n integers. In other words, for \( i = 1, 2, \ldots, n \):

\[
s(n+1, n+1-l) = \sum_{1 \leq i_1 < i_2 < \ldots < i_l \leq n} i_1 i_2 \ldots i_l.
\]

Differently formulated, the unsigned Stirling number of the first kind \( s(n, k) \) appears here as the sum of all products of \( n-k \) different integers taken from \( [n-1] = \{1, 2, \ldots, n-1\} \). (There are \( \binom{n-1}{k-1} \) such products.)

For instance, \( s(6, 2) = s(6, 2) = 1.2.3.4 + 1.2.3.5 + 1.2.4.5 + 1.3.4.5 + 2.3.4.5 = 274 \).

This is clear from [5h], or if one prefers:

\[
s(n+1, n+1-l) = \sum_{0 \leq i_1 < i_2 < \ldots < i_l \leq n} i_1 i_2 \ldots i_l.
\]

(For generalizations, see [Toscano, 1939], [Storchi, 1948]).

5.6. Recurrence relations between the \( s(n, k) \)

**Theorem A.** The Stirling numbers of the first kind \( s(n, k) \) satisfy the 'triangular' recurrence relation:

\[
s(n, k) = s(n-1, k-1) - (n-1)s(n-1, k), \quad n, k \geq 1;
\]

\[
s(n, 0) = s(0, k) = 0, \quad \text{except} \quad s(0, 0) = 1.
\]

For the unsigned numbers, this can be written

\[
s(n, k) = s(n-1, k-1) + (n-1)s(n-1, k),
\]

This is a means for a quick computation of the first values of the \( s(n, k) \) (see table on p. 310 and Exercise 16, p. 226); particularly:

\[
s(n, 1) = (-1)^{n-1}(n-1)!,
\]

\[
s(n, n-1) = -\binom{n}{2}, \quad s(n, n) = 1.
\]

Equate the coefficients of \( x^k \) in the first and last member of [6c]:

\[
\sum_k s(n, k)x^k = (x)_n = (x - (n - 1))(x)_{n-1} = (x - (n - 1))\sum_h s(n-1, h)x^h.
\]

**Theorem B.** The \( s(n, k) \) satisfy the 'vertical' recurrence relations:

\[
\sum_{k \geq 1} s(n, k)x^k = (x)_n = (x - (n - 1))(x)_{n-1} = (x - (n - 1))\sum_h s(n-1, h)x^h.
\]

\[
s(n+1, k+1) = \sum_{k \geq 1} (-1)^{n-1} \left( \frac{n}{k} \right) s(l, k-1),
\]

\[
s(n+1, k+1) = \sum_{k \geq 1} (-1)^{n-1} (l+1)(l+2) \cdots (n)s(l, k).
\]

**Theorem C.** The \( s(n, k) \) satisfy the 'horizontal' recurrence relations ([Lagrange, 1771]):

\[
(n-k)s(n, k) = \sum_{k \geq 1} (-1)^{n-k} \left( \frac{l}{k-1} \right) s(n, l)
\]

\[
s(n, k) = \sum_{k \geq 1} s(n+1, l+1)n^{l-k}.
\]

Equate the coefficients of \( x^k \) in the expressions to the right of (*) and (**):

\[
x(x-1)_n = x\sum_l s(n, l)(x-1)^l
\]

\[
\sum_{k, l} (-1)^{n-k}s(n, k)\left( \frac{l}{k} \right) x^{k+1} = (x - n)(x)_n
\]

\[
\sum_j s(n, j)x^{j+1} - n\sum_j s(n, j)x^j.
\]

For [6g], equate the coefficients of \( u^{n-k} \) in \( \Psi_{n-1} = (1 - (n-1)u)^{-1}\Psi_n \).

[5g] (p. 213).

Figure 35 shows the diagrams of the recurrence relations established
in the preceding section. (See p. 12. Analogous diagrams hold evidently as well for the recurrence relations of pp. 208 and 209.)

5.7. THE VALUES OF \( s(n, k) \)

According to \([1b]\) (p. 204) the Stirling number of the second kind \( S(n, k) \) can be expressed as a single summation of elementary terms, that is, which are themselves products and quotients of factorials and powers. There does not exist an analogous formula for the numbers of the first kind, the 'shortest formula' \([7a, a']\) below being a double summation of elementary terms. Shortwise, we will say that \( S(n, k) \) is of rank one and that \( s(n, k) \) is of rank two.

THEOREM A. ([Schômilch 1852]). The 'exact' value of \( s(n, k) \) is:

\[
\begin{align*}
[7a] \quad s(n, k) &= \sum_{0 \leq k \leq n} (-1)^k \binom{n-k}{k} \binom{2n-k}{n-k} S(n-k+h, h) \\
[7a'] &= \sum_{0 \leq j \leq h \leq n-k} (-1)^{j+n} \binom{h}{j} \binom{2n-k}{n-k} \binom{h-j}{k} \binom{h-j}{n-k-h} \frac{1}{h!}.
\end{align*}
\]

We use the Lagrange formula (p. 148). Let \( f(t) = e^t - 1 \) and its inverse function \( f^{-1}(t) = \log(1+t) \). We get, by \([3b, c]\) (p. 134), tabulated on p. 307) and \( \zeta_s(s) = -\sum_{j=1}^{n} j^{-s} \).

THEOREM B. We have:

\[
[7b] \quad s(n+1, k+1) = \frac{n!}{k!} \frac{1}{\zeta_s(1)} = \frac{n!}{k!} \sum_{j=1}^{k}(1+j)^{-s}.
\]

where \( \zeta_s(s) \) stands for the Bell polynomial (complete exponential, \([3b, c]\) (p. 134), tabulated on p. 307) and \( \zeta_s(s) = -\sum_{j=1}^{n} j^{-s} \).

\( \Box \)

In fact, by \([5j]\) (p. 214) for \(*\):

\[
\sum_k s(n+1, k+1) x^k = n! (1+x) \left( 1 + \frac{x}{2} \right) \cdots \left( 1 + \frac{x}{n} \right) = n! \exp \left( \sum_{j=1}^{n} \log(1+j^{-1}) \right) = n! \exp \left( \sum_{j=1}^{n} \frac{(-1)^{j-1} x^{j-1}}{j} \right).
\]

and then we apply definition \([3c]\) (p. 307) of the \( \zeta_s(s) \).

(There is an analogous formula for each elementary symmetric function Exercise 9, (4) p. 158. See also Exercises 16, p. 226, and 9, p. 293.)

Thus:

\[
s(n+1, 2) = n! \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n} \right) = n! H_n,
\]

where \( H_n \) denotes the harmonic number.

\[
s(n+1, 3) = \frac{n!}{2} \left( H^2_n - \left( 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \right) \right),
\]

\[
s(n+1, 4) = \frac{n!}{6} \left( H^3_n - 3H_n \left( 1 + \frac{1}{2^3} + \cdots + \frac{1}{n^3} \right) + 2 \left( 1 + \frac{1}{2^4} + \cdots + \frac{1}{n^4} \right) \right).
\]
5.8. Congruence problems

It is interesting to know in advance or to discover some congruences in any table of a sequence of combinatorial integers. This is a rapid way of checking computations, and an attractive connection between Combinatorial Analysis and Number Theory. We show two typical examples in this matter.

Let two polynomials be given:
\[ f(x) = \sum_k a_k x^k, \quad g(x) = \sum_k b_k x^k \]
with integer coefficients, \( a_k, b_k \in \mathbb{Z} \). We often write, when \( a_k \equiv b_k \pmod{m} \) for all \( k \):
\[ [8a] \quad f(x) \equiv g(x) \pmod{m} \]
and we say 'f congruent g modulo m'.

**Theorem A.** (Lagrange). For each prime \( p \), we have in the sense of \([8a]\):
\[ [8b] \quad (x)_p := x(x - 1) \cdots (x - p + 1) \equiv x^p - x \pmod{p}. \]
In other words, the Stirling numbers of the first kind satisfy:
\[ [8c] \quad s(p, k) \equiv 0 \pmod{p}, \quad \text{except } s(p, p) = 1, \quad \text{and (Wilson theorem)} \]
\[ [8d] \quad s(p, 1) = (p - 1)! \equiv -1 \pmod{p}. \]

For \( p \) fixed, we argue by induction, on \( k \) decreasing from \( p - 1 \). By \([6b]\) (p. 214), for (\(*\)), and Theorem C (p. 15) for (\(**\)), \([8c]\) is true when \( k = p - 1 \):
\[ s(p, p-1) \equiv -\binom{p}{2} \equiv 0 \pmod{p}. \]
Now, by \([6f]\) (p. 215):
\[ [8e] \quad (p-k)s(p, k) = \sum_{k+1 \leq i \leq p} (-1)^{p-k-1} \binom{p}{i} s(p, i). \]
Assume that \([8c]\) is true, thus, \( s(p, l) \equiv 0 \pmod{p} \) for \( 3 \leq k + 1 \leq l \leq p - 1 \). Then, \([8e]\) implies, by Theorem C (p. 14) for (\(*\)):
\[ -k s(p, k) \equiv (-1)^{p-k} \binom{p}{k-1} s(p, p) \equiv 0 \pmod{p}, \]
from which \([8c]\) follows, since \( 2 \leq k \leq p - 2 \).

**Consequence (Fermat theorem).** For all integers \( a \geq 0 \), and each prime number \( p \),
\[ [8f] \quad a^p \equiv a \pmod{p}. \]
Put \( x = a \) in \([8b]\), then \( (a)_p \equiv 0 \pmod{p} \), because, among \( p \) consecutive integers, at least one is a multiple of \( p \).

**Theorem B.** For each prime number \( p \), the Stirling numbers of the second kind satisfy:
\[ [8g] \quad S(p, k) \equiv 0 \pmod{p}, \quad \text{except } S(p, 1) = S(p, p) = 1. \]

**Supplement and Exercises**

1. **Banners and chromatic polynomials.** (1) Show that the number \( d(n, k) \) of banners with \( n \) vertical bands and \( k \) colours, two adjacent bands of different colour, equals \( k! S(n-1, k-1) \). (2) Moreover, for every tree \( T \) over \( N \), \( |N| = n, d(n, k) \) is also the number of colourings of the \( n \) nodes with \( k \) colours such that two adjacent nodes have a different colour. (Compare with Exercise 1, p. 198.) (3) More generally, considering a
graph $G$ with $n$ nodes and introducing the number $d(G, k)$ of colourings of these nodes with $k$ colours, having the preceding property, show that the chromatic polynomial (of p. 179) satisfies: $P_G(\lambda) = \sum_{k=0}^{\lambda} d(G, k) \times \binom{\lambda}{k}$. *(4) What is the number of checkerboards of dimensions $(m \times n)$ with $k$ colours? (Two squares with a side in common must be coloured differently.)

2. Lie derivative and operational calculus. Let $\lambda(t)$ be a formal series. We define the operator $\lambda D$ (Lie derivative) by $(\lambda D)f := \lambda f' + \lambda f$, where $D$ is the usual derivation (p. 41). Similarly, $(\lambda D)f := \lambda f(\lambda f) = \lambda f' + \lambda f$. (1) $(tD)^n = \sum_{i=1}^{n} S(n, i) t^1 D^i$ and $(D^t)^n = \sum_{i=0}^{n-1} S(n+1, i+1) t^1 D^i$. (2) $(e^B^D)^n = \sum_{i=1}^{n} S(n, i) b^i D^i$. (3) $(e^{a+1} D)^n = \alpha^n \sum_{i=1}^{n} P_n(a) t^i D^i$, where $\sum_{a \geq 1} P_n(a) t^{1/n!} = (1 - \alpha)^{-1/q-1}$. *(4) Find an explicit formula for $(tD)^n$ and $(D^t)^n$ [(Concrete, 1973)]. *(5) The following result of Pournchet shows that this problem is closely connected with the Faà di Bruno formula:

\[
\left( \frac{d}{dx} \right)^n f(x) = \frac{d^n}{dx^n} f(x(w)), \quad \text{where} \quad \frac{dx}{dw} - \lambda(x), \quad x = x(w).
\]

Apply this method to prove: $(x \log x D)^n = \sum_{k=1}^{n} S(n, k) S(l, k)$ ($\log x)^k D^k$.

3. The lattice of the partitions of a set. Let be given two partitions $\mathcal{F}$, $\mathcal{F}$ of a set $N$. Then we say that $\mathcal{F}$ is finer than $\mathcal{F}$ or that $\mathcal{F}$ is a subpartition of $\mathcal{F}$, denoted by $\mathcal{F} \subseteq \mathcal{F}$, if and only if each block of $\mathcal{F}$ is contained in a block of $\mathcal{F}$. Show that this order relation on the set of partitions of $N$ makes it into a lattice (Definition D, p. 59).

4. Bernoulli and Stirling numbers and sums of powers. We write the GF of the Bernoulli numbers $B_n$ [14a] (p. 48), in the form:

\[
\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{t}{e^t - 1} - \frac{1}{e^t - 1}.
\]

Show that $B_n = \sum_{k=0}^{n-1} (-1)^k S(n, k) (k+1)$. Use this to obtain the value of $B_n$ expressed as a double sum. Show also, by substituting $u = e^t - 1$ into $[\#]$, that $\sum_{k=0}^{n} s(n, k) B_k = (-1)^n n! / (n+1)$. Verify the formula $B_n = \sum_{k=0}^{n} (-1)^{n-k} Z(j, k) / (j+1) / (j+1)$, where $Z(j, k) = 1^{j+2} 2^{j+3} + \cdots + j^k$ [(Bergmann, 1967) and p. 155. See also [Gould, 1972] which gives other explicit formulas for the Bernoulli numbers]. Show that $Z(n, k) = \sum_{j=1}^{n+1} (-1)^{j-1} (j-1)! S(r+1, j) \binom{n}{j}$.

5. A transformation of formal series. For each integer $k \geq 0$, let $T_k$ be the transformation of formal series defined by: $f = \sum_{n=0}^{\infty} a_n t^n \rightarrow T_k f = \sum_{n=0}^{\infty} a_n t^n$. *(1) Show that $T_k f = \sum_{n=0}^{\infty} S(n, k) t^{n+1} D^k$ (is the differentiation operator, p. 41). *(2) Deduce from this the value of $\sum_{n=0}^{\infty} n^t n_1$ in the form of a rational fraction, and also that of $\sum_{n=0}^{\infty} n^t n_1$. *(3) Furthermore, with the Eulerian numbers $A(k, h)$ (pp. 51 and 242) we have $\Phi_k(t) := (1 - t)^{k+1} \sum_{n=0}^{\infty} n^t n_1 \rightarrow \sum_{k=1}^{n} A(k, h) / t^k$. *(4) Express $\sum_{n=0}^{\infty} n^t n_1$ in the form of a product of $e^t$ with a polynomial. *(5) Solve analogous problems for $\sum_{n=0}^{\infty} n^t(t+t_1)n_1$ (and $\sum_{n=0}^{\infty} n^t(t+t_1)n_1$), where $x$ is a complex number. *(6) Study the transformation $T_k c$, with $c$ a given integer $\geq 0$, such that $T_k c f := \sum_{n=0}^{n+c} (n+c)! a_n t^n$.

6. The Taylor-Newton formula. For each polynomial $P(x)$ we have $D$ is the difference operator defined on p. 13):

\[
P'(x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} A^k P(a) = (1 + A)^x P(0).
\]

More generally, let be given a sequence $a_0, a_1, a_2, \ldots$ of different complex numbers, $f$ a formal series (with complex coefficients) and $t, x$ two indeterminates. We put $f'(x) = (x - a_0) (x - a_1) \cdots (x - a_{k-1})$ and $\mathcal{A}_k = 0 \ldots 0 \mathcal{A}_0$ for $\mathcal{A}_k$ and $\mathcal{A}_0$. Prove then the multiplication formula:

\[
f'(x) = \sum_{k=0}^{\infty} f'(x) \sum_{j=0}^{\infty} A(j, k) a_j t^j.
\]

Use this to recover the formulas of Exercise 29 (p. 167).

7. Associated Stirling numbers of the second kind. For $r$ integer $\geq 1$, let $S_r(n, k)$ the number of partitions of the set $N$, $|N|=n$, into $k$ blocks, all of cardinality $\geq r$. We call this number the $r$-associated Stirling number
of the second kind. In particular, \( S_1(n, k) = S(n, k) \). Then we have the GF:

\[
\sum_{n, k \geq 0} S_r(n, k) \frac{t^n}{n!} = \exp \left\{ u \left( \frac{t^r}{r!} + \frac{t^{r+1}}{(r+1)!} + \cdots \right) \right\},
\]

and the 'triangular' recurrence relations:

\[
S_r(n+1, k) = kS_r(n, k) + \sum_{r_1 + \cdots + r_k = r} S_r(n - r_1, k - 1) \cdots S_r(n - r_k, k - 1).
\]

Moreover, \( S_2(n, k) \equiv 0 \pmod{1, 3, 5, \ldots, (2k-1)} \) and, for \( l \geq 1 \), 

\[
(-1)^l l! = \sum_{m=1}^{r} (-1)^m S_2(l + m, m).
\]

The first values of \( S_r(n, k) \) are:

\[
\begin{array}{cccccccc}
\hline
k \backslash n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 3 & 10 & 25 & 56 & 119 & 246 & 501 \\
3 & 1 & 15 & 105 & 490 & 1918 & 6825 & 22935 & 74316 \\
4 & 1 & 105 & 1260 & 9450 & 56980 & 302995 & 10395 & 34459425 \\
5 & 1 & 945 & 17325 & 190575 & 1487200 & 6914908 & 30950920 & 135135 \\
6 & 1 & 10395 & 235092 & 731731 & 2252341 & 1487200 & 6914908 & 30950920 & 135135 \\
7 & 1 & 111111 & 32751 & 65518 & 131053 & 34459425 & 135135 & 4729725 & 94594500 \\
8 & 1 & 1111111 & 32751 & 65518 & 131053 & 34459425 & 135135 & 4729725 & 94594500 & 34459425 \\
9 & 1 & 11111111 & 32751 & 65518 & 131053 & 34459425 & 135135 & 4729725 & 94594500 & 34459425 & 11111111 \\
\hline
\end{array}
\]

Let \( P_r(t) = \sum_{k=0}^{r} t^k / k! \). Use the \( S_r(n, k) \) to expand \( (P_r(t))^n \), \( P_r(t) \), \( P_r(u) \) and \( \log(P_r(t)) \).

8. Distributions of balls in boxes. The number of distributions of \( n \) balls into \( k \) boxes equals: (1) \( k^n \) if all balls and all boxes are different; \( k! S(n, k) \) if no box is allowed to be empty. (2) \( \binom{n+k-1}{n} \) if the balls are indistinguishable, and all the boxes different; \( \binom{n-1}{k-1} \) if, moreover, no box is allowed to be empty (Theorem C., p. 15). (3) Suppose the boxes are all different, and the balls of equal size, but painted in different colours. Balls of the same colour are supposed to be not distinguishable. In this way we define a partition of the set \( N \) of balls. If there are in this partition \( c_i \), \( i = 1, 2, 3, \ldots \), then the number of distributions is equal to \( \prod_{i=1}^{c_i} \binom{k}{i} \binom{k+1}{2} \binom{k+2}{3} \cdots c_i + 2c_2 + \cdots = n \) [use (2)]. (4) What do we get for all the preceding answers when the boxes and balls are put in rows? (For all these problems, see especially [*MacMahon, 1915-16]. Good information is also found in [*Riordan, 1958], pp. 90-106.)

9. Return to the Bell polynomials. Application to rational fractions. The exponential partial Bell polynomials \( B_{n,k} \) are a generalization of the Stirling numbers, because \( B_{n,k}(1, 1, \ldots) = S(n, k) \), [3z] (p. 135). (1) Let \( a_1, a_2, \ldots \) be integers \( \geq 0 \). Show that \( B_{n,k}(a_1, a_2, \ldots) \) equals the number of partitions of \( N, \mid N \mid = n, \) into \( k \) blocks, the \( i \)-blocks being painted with colours taken from a stock \( A_i \), given in advance, and with \( a_i \) colours in the stock \( A_i \), \( i = 1, 2, 3, \ldots \). (It is not compulsory to use all colours of each stock!) (2) We denote the value of the \( n \)-th derivative in the point \( x = a \) of \( F(x) \) [or \( G(x) \)] by \( f_{1:a}, \ldots; g_{0:a} = F(a), G(a) \). Suppose that \( x = a \) is a multiple root of order \( k \) of \( G(x) = 0 \), and that \( F(x)/G(x) \) has the singular part \( \sum_{p=1}^{k} y_p (x-a)^{-p} \), Show that the coefficients \( y_p \) equal:

\[
\sum_{0 \leq p \leq k-1} \frac{(-1)^p}{k-p} B_{k-p-1} \binom{g_{k-p}}{1} \binom{g_{k+2}}{2} \cdots.
\]

(For \( k = 1 \) we recover \( y = f_{0:a} / g_{1:a} = F(a) / G'(a) \), that is the residue of \( F/G \) when \( x = a \) is a simple pole.) (3) Now take \( F \) and \( G \) to be polynomials, \( G = \prod_{i=1}^{n} (x-a_i)^{a_i} \), with all different \( a_i \). Express the \( y_{p:a} \) by an 'exact' formula of rank \( \leq n - 2 \).

10. The Schröder problem. ([Schröder, 1870]. See also [Carlitz, Riordan, 1955], [Comtet, 1970], [Knölker, 1951]). Let \( N \) be a finite set, \( \mid N \mid = n \), and let us use the name 'Schröder system' for any system (of blocks of \( N \))
\[ S \subset \Psi'(N) \text{ such that: } (a) \text{ Every 1-block of } N \text{ belongs to it: } \Psi_1(N) \subset S \]

\[ (b) N \text{ does not belong to it: } N \not\in S \]

\[ (\gamma) \text{ } R, B' \in S \Rightarrow R \subset B' \text{ or } B' \subset R \text{ or } B \cap B' = \emptyset. \]

We denote the family of all Schröder systems of \( N \) by \( s(N) \), and the problem is now to compute its cardinal \( s_* = |s(N)| \). (1)

Let the number \( k_i \) of maximal \( i \)-blocks be fixed, \( i \in [n-1] \). (Maximal block is a block contained in no other.) Then we have:

\[ k_1 1 2k_2 + \cdots + (n-1)k_{n-1} - n, \]

and the number of the corresponding \( S \in s(N) \) equals: \( n!k_1k_2^2 \cdots (1)^{-k_1}(21)^{-k_2}(k_1)^{-1}(k_2)^{-1} \cdots ; s_1 = 1 \)

(2) Observe that the condition [a] is equivalent to the two conditions \( k_1 + 2k_2 + \cdots + nk_n = n, k_1 + k_2 + \cdots + k_n \geq 2 \). Show that the GF \( y = \sum_{n \geq 0} s_n t^n/n! \) satisfies:

\[ e^t - 2y - 1 + t = 0, \]

(3) We have \( s_n = \sum_{k=0}^{n-2} S_2(n+k, k+1) \), codiagonal sums of the associated Stirling numbers of Exercise 7 (p. 221). So, \( s_4 = 1 + 10 + 15 = 26 \). Hence the table of values:

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_n )</td>
<td>1</td>
<td>1</td>
<td>4</td>
<td>26</td>
<td>236</td>
<td>2752</td>
<td>39208</td>
<td>660032</td>
<td>12818912</td>
<td>282137824</td>
</tr>
</tbody>
</table>

(4) \( s_p \equiv 1 \text{ (mod p), for p prime.} \)

(5) \( s_n = \sum_{k \geq 1} 2^{-n-k} S(n+k-1, k) \) (style Dobinski). (6) Explicitly,

\[ S_n = \sum_{1 \leq j+1 \leq k \leq n} (-1)^{j+k+1} \frac{1}{k!} \binom{n+1}{k} \binom{k}{j} \binom{n+k-1}{n+k-j} (k-j)^{j+k-1}. \]

(7) Asymptotically,

\[ s_n \approx \frac{1}{2} \sqrt{\frac{A}{\pi n}} \frac{(n-1)!}{n^a} \frac{1 + \sum d_i}{n^{a+1}}, \quad n \to \infty, \]

where \( A = 2 \log 2 - 1 = 0.386294 \ldots \) and \( d_i \) are polynomials in \( A \):

\[ d_1 = (9 - A)/24, \quad d_2 = (225 - 90A + A^2)/192, \ldots. \]

11. Congruences of the (Bell) number of partition \( \omega(n) \). Let \( p \) be a prime number. Modulo \( p \), we have \( \omega(p) \equiv 2, \omega(p+1) \equiv 3 \) and, more generally, \( \omega(p^r + h) \equiv \omega(h) + \omega(h+1) \). Modulo \( p^k \), we have \( \omega(2p) - 2\omega(p+1) - 2\omega(p) + p + 5 \equiv 0 \) ([Touchard, 1956]).

12. Generalization of \( \sum \binom{n}{k}^2 = \binom{2n}{n} \). Let \( P_{n,r}(z) = \sum_{k=0}^{n} k^r \binom{n}{k}^2 2^{n-k} \), where \( r \) is integer \( \geq 0 \). Use Exercise 5 (1) (p. 221) to show that

\[ P_{n,r}(z) = \sum_S S(r, q) (n)_q \binom{1+z}{1}^q \binom{1+i}{1}^{q-r}. \]

Thus,

\[ A(n, r) := P_{n,r}(1) = \sum_{k=0}^{n} k^r \binom{n}{k}^2 = \sum_S S(r, q) (n)_q \binom{2n-q}{n}. \]

Particularly,

\[ A(n, 0) = \binom{2n}{n}, \quad A(n, 1) = \binom{2n-2}{n-1}, \quad A(n, 2) = -n^2 \binom{2n-2}{n-1}, \]

\[ A(n, 3) = n \binom{n+1}{2} \binom{2n-2}{n-1}. \]

Similarly,

\[ \sum_{k=0}^{n} (-1)^{r} k^r \binom{n}{k} = \sum_{l \geq j+q=n} (-1)^{j} S(q, r)(n)_q \binom{n-q}{i} \binom{q}{j}. \]

13. A 'universal' generating function. The following solves, for partitions of a set, a problem analogous to the problem for partitions of integers, which is solved by Theorem B (p. 98). Let \( \mathcal{U} \) be an infinite matrix consisting of 0 and 1, \( \mathcal{U} = [\sigma_{i,j}], i \geq 1, j \geq 0, \sigma_{i,j} = 0 \text{ or } 1 \). Let \( s(n | k, \mathcal{U}) \) be the number of partitions of a set \( N \) into \( k \) blocks such that the number of blocks of size (= cardinal number) \( i \) equals to one of the integers \( j \geq 0 \) for which \( \sigma_{i,j} = 1 \). Then we have the 'universal' GF:

\[ \sum_{a, k \geq 0} s(n | k, \mathcal{U}) u^k k^n/n! = \prod_{i \geq 1} \left\{ \sum_{j \geq 0} \sigma_{i,j} \binom{u^i}{j!} \right\}. \]

In particular we obtain the following table of GF, where \(* \) means no condition \((n-1) \text{ provides the 'total' GF):
Number of blocks | Size of each block | GF by 'number of blocks' | 'Total' GF
--- | --- | --- | ---
* | * | \(\exp(\alpha(e^t-1))\) | \(\exp(\alpha(e^t-1))\)
* | odd | \(\exp(\alpha(ch^t-1))\) | \(\exp(\alpha(ch^t-1))\)
* | even | \(\exp(\alpha(ch^t-1))\) | \(\exp(\alpha(ch^t-1))\)
even | * | \(\exp(\alpha(ch^t-1))\) | \(\exp(\alpha(ch^t-1))\)
odd | even | \(\exp(\alpha(ch^t-1))\) | \(\exp(\alpha(ch^t-1))\)
even | odd | \(\exp(\alpha(ch^t-1))\) | \(\exp(\alpha(ch^t-1))\)
odd | even | \(\exp(\alpha(ch^t-1))\) | \(\exp(\alpha(ch^t-1))\)
even | even | \(\exp(\alpha(ch^t-1))\) | \(\exp(\alpha(ch^t-1))\)

14. 'Stackings' of \(x\). Let \(f_1(x)\) be the sequence of functions defined by \(f_1(x) = x, f_2(x) = x^2, \ldots, f_i(x) = x^{i-1}, i \geq 2\). Determine and study the coefficients of the expansion \(f_i(x) = \sum_{r=0}^{\infty} a_i(r, q) x^r \log^q x\). (See also p. 139.)

*15. The number of 'connected' \(n\)-relations. Let \(p\) and \(q\) be two integers \(\geq 1\). A relation \(\mathcal{R} \subseteq [p] \times [q]\) is called 'connected' if \(pr, \mathcal{R} - [p]\), \(\mathcal{R} = [q]\) (p. 59), and if any two points of \(\mathcal{R}\) can be connected by a polygonal path with unit sides in horizontal or vertical direction, all whose vertices are in \(\mathcal{R}\). We say also that \(\mathcal{R}\) is \((p \times q)\)-animal. Thus, in Figure 36, (I) is an animal, but (II) is not. Compute or estimate the number \(A(n; p, q)\) of the \(A\) such that \(|A| = n\), also called 'n-ominos' (This term is taken from [Golomb, 1966]. For an approach to this problem, see [Kreweras, 1969] and [Read, 1962a]). Analogous question for dimension \(d \geq 3\), \(A \subseteq [p_1] \times [p_2] \times \cdots \times [p_d]\).

16. Values of \(S(n, n-a)\) and \(s(n, n-a)\). (1) We have \(S(n, n-a) = \sum_{j=\#a+1}^{n} \binom{n}{j} S_2(j, j-a)\). \(S_2\) as defined in Exercise 7 (p. 221). Thus, \(S(n, n) = 1\),

\[
S(n, n-1) = \binom{n}{2}, S(n, n-2) = \binom{n}{3} + 3 \binom{n}{4} = \frac{n}{3} (3n-5),
\]

\[
S(n, n-3) = \binom{n}{4} + 10 \binom{n}{5} + 15 \binom{n}{6} = \frac{5}{4} \binom{n}{4} (n^2 - 5n + 6).
\]

(2) Similarly, we have \(s(n, n-a) = \sum_{j=\#a+1}^{n} \binom{n}{j} s_2(j, j-a)\), where the \(s_2\) are defined by \(\sum_{j=\#a+1}^{n} s_2(n, k) t^a u^{n-1} e^{-u} (1 + t)^y\). (Exercise 7, p. 256; Exercise 20, p. 295). Thus, \(s(n, n) = 1, s(n, n-1) = -\binom{n}{2}, s(n, n-2) = 2 \binom{n}{3} + 3 \binom{n}{4} = \binom{n}{4} (3n-1), s(n, n-3) = -6 \binom{n}{4} - 20 \binom{n}{5} - 15 \binom{n}{6} = -\frac{5}{4} \binom{n}{4} (15n^3 - 30n^2 + 5n + 2)\). (Other 'exact' formulas in [Mitrinović, 1960, 1961, 1962]. See also Exercise 9, p. 293.)

17. Stirling numbers and Vandermonde determinants. The value of the unsigned number of the first kind \(S(n+1, k)\) is the quotient of the \(n\)-th order determinant obtained by omitting the \(k\)-th column of the matrix

\[
\begin{vmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & 2 & 2 & \cdots & 2 \\
1 & 3 & 3 & \cdots & 3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & n & n & \cdots & n
\end{vmatrix}
\]

by \(1!2!\cdots(n-1)!\). The number of the second kind \(S(n, k)\) can be expressed using a determinant of order \(k\):

\[
k! S(n, k) = \begin{vmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & 2 & 2 & \cdots & 2 \\
1 & 3 & 3 & \cdots & 3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & k & k & \cdots & k
\end{vmatrix}
\]

18. Generalized Bernoulli numbers. These are the numbers \(B_n^{(r)}\) defined for every complex number \(r\):

\[
\left( \frac{t}{e^t-1} \right)^r = \sum_{n=0}^{\infty} B_{n}^{(r)} \frac{t^n}{n!}.
\]
Evidently $B_t^{(0)} = B_t$ (14a) (p. 48). Show (with [5h] p. 142) that $B_t^{(n)} = \sum_{j=0}^{n-1} \binom{n+1}{j+1} B_{t-j}^{(n-1)}$. Moreover, for all pairs of integers $(n, p)$ such that $0 \leq n \leq p - 1$ we have $B_t^{(p)} = \binom{p-1}{n} s(p, p-n)$. Besides, $B_t^{(p)} = \sum_{n=0}^{p} B_t^{(n)} \chi(n, 1, \ldots)$, by [5d] (p. 141): $B_0^{(0)} = 1$, $B_1^{(0)} = -r/2$, $B_2^{(0)} = \frac{1}{12} r(3r-1)$, $B_3^{(0)} = -\frac{1}{4} r^2 (r-1)$, and so on. Finally, determine an 'exact' formula of minimal rank for $B_t^{(n)}$ (p. 216).

19. Diagonal differences. Show that $A_t^{(2)} S(k, k+j) = A_t^{(2)} S(k, k+j) = \prod_{k=1}^{j} (2k-1)$. 

20. The number of 'Fubini formulas'. Let $a_n$ be the number of possible ways to write the Fubini formula (111] p. 34) for a summation of integration of order $m$. Evidently, $a_1 = 1$, $a_2 = 3$, $a_3 = 13$, because $\sum_{c_1, c_2, c_3} = \sum_{c_2, (\sum_{c_1, c_3})} = \sum_{c_1, (\sum_{c_2, c_3})} = \sum_{c_1, c_2, c_3} (\sum_{c_1, c_2, c_3}) = \sum_{c_1, c_2, c_3} (\sum_{c_2, c_3}) = \sum_{c_1, c_2} (\sum_{c_2, c_3}) = \sum_{c_1, c_2} (\sum_{c_1, c_3}) = \sum_{c_1, c_2, c_3} (\sum_{c_1, c_2, c_3}) = \sum_{c_1, c_2} (\sum_{c_1, c_3}) = \sum_{c_1, c_2} (\sum_{c_2, c_3}).$

Show that $a_n = \sum_{m=1}^{n+1} k! S(m, k)$ and that $\sum_{m>0} a_n m^m/m! = (2e-1)^{-1}$. 

Moreover, $a_n = \sum A(n, k) 2^{k-1}$, as a function of the Eulerian numbers of p. 51 or 242, and $a_n = \|m! (\ln 2)^{-n} 2^{-1} \|$ (notation [6f], p. 110).

21. A beautiful determinant. Let $s$ be the unsigned Stirling numbers of the first kind (p. 213). Then,

$$s(n+1, 1) = s(n+2, 2) \ldots s(n, k) = (n!)^k.$$ 

22. Inversion of $y e^y$ and $y \log y$ in a neighbourhood of infinity ([Comtet, 1970]). The equations $y e^y = x$ and $y \log y = x$, where $a$ and $\beta$ are constants $\geq 0$, have solutions $y = \Phi_a(x)$ and $y = \Psi_\beta(x)$ that tend to infinity for $x$ tending to infinity. Then, with $L_1 := \log x$ and $L_2 := \log \log x$, we have:

$$\Phi_a(x) = L_1 - \alpha L_2 + \sum_{n>1} \left\{ \left( \frac{-a}{n} \right)^{n+1} \sum_{m=1}^{n} s(n, n-m+1) \frac{L_m^n}{m!} \right\},$$

$$\Psi_\beta(x) = \frac{x}{L_1} \left\{ 1 + \sum_{n>1} \left( \frac{-\beta}{n} \right)^{n+1} \sum_{m=1}^{n} \frac{(-L_2)^n}{m!} Q_{n,m}^n(\beta) \right\},$$

the polynomial $Q_{n,m}^n(\beta)$ being $\sum_{k=1}^{n-m+k-1} \binom{n-m+k-1}{n-m} s(n, n-m+k) \beta^k$.

23. Congruences of the Stirling numbers. Let $p$ be prime. We denote 'a divides b' by a | b. (1) $p^2 \mid s(p, 2h)$ for $2 \leq 2h \leq p-3$ and $p \geq 5$ (Nielsen). Particularly, the numerator of the harmonic number $H_{p-1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{(p-1)}$ is divisible by $p^2$. (2) $p \mid S(p+1, k)$ for $3 \leq k \leq p$ and $p \mid (S(p+1, 2) - 1)$.

24. An asymptotic expansion for the sum of factorials. If $n \to \infty$, we have:

$$\frac{1}{n!} \sum_{k=1}^{n} k! \approx 1 + \sum_{k=1}^{n} \omega(k) = 1 + \frac{1}{n} + \frac{1}{n^2} + \frac{2}{n^3} + \frac{5}{n^4} + \cdots.$$

25. The number of topologies on a set of $n$ elements. This number $t_n$ equals $\sum_{k=1}^{n} S(n, k) d_k$, the $d_k$ being the number of order relations defined on p. 60 ([Comtet, 1966]).
PERMUTATIONS

6.1. The symmetric group

We recall that a permutation \( \sigma \) of a finite set \( N \), \(|N| = n \), is a bijection of \( N \) onto itself.

Actually, as \( N \) is finite, we could as well have said 'surjection' or 'injection' instead of 'bijection'.

A permutation \( \sigma \) can be represented by writing the elements of the set \( N \) on a top row, and then underneath each element its image under the mapping \( \sigma \). Thus \( \begin{pmatrix} a & b & c & d & e & f & g \\ a & c & e & d & b & f & g \end{pmatrix} \) represents a \( \sigma \in \mathfrak{S}(N) \), where \( N = \{a, b, c, d, e, f, g\} \), \( \sigma(a) = c, \sigma(b) = a, \sigma(c) = e, \sigma(d) = d, \sigma(e) = b, \sigma(f) = g, \sigma(g) = f \).

Another way of representing \( \sigma \) consists of associating with it a digraph \( D \) (p. 67), where it is understood that an arc \( xy \) is drawn if and only if \( y = \sigma(x) \), \( y \neq x \). Figure 37 corresponds in this way with the above permutation.

One can also represent \( \sigma \) by a relational lattice, as on p. 58. Then Figure 38 corresponds with the permutation of Figure 37. Clearly a binary relation on \( N \) is associated with a permutation in this way if and only if all its horizontal and vertical sections have one element.

Finally, \( \sigma \) can be represented by a square matrix, say \( B = [b_{i,j}] \), defined by \( b_{i,j} = 1 \) if \( j = \sigma(i) \), and \( b_{i,j} = 0 \) otherwise. Such a matrix is called a permutation matrix.

We denote the group of permutations of \( N \) (with composition of maps as operation) by \( \mathfrak{S}(N) \). This group is also called the symmetric group of \( N \). The unit element of this group is the identity permutation, denoted by \( e : \forall x \in N, e(x) = x \). Evidently, \( |\mathfrak{S}(N)| = n! \) (p. 7).

We recall some notions about permutations \( \sigma \in \mathfrak{S}(N) \).

The orbit of \( x \in N \) for a permutation \( \sigma \) is the subset of \( N \) consisting of the points \( x, \sigma(x), \sigma^2(x), \ldots, \sigma^{k-1}(x) \), where \( k \), the length of the orbit, is the smallest integer \( \geq 1 \) such that \( \sigma^k(x) = x \). If \( k = 1 \), \( \sigma(x) = x \), then \( x \) is a fixed point of \( \sigma \) (See p. 180).

Let \( x_1, x_2, \ldots, x_k \) be \( k \) different points of \( N \), \( 1 \leq k \leq n \). The cycle \( \gamma = (x_1, x_2, \ldots, x_k) \) is the following permutation: \( \gamma(x_1) = x_2, \gamma(x_2) = x_3, \ldots, \gamma(x_{k-1}) = x_k, \gamma(x_k) = x_1 \) and \( \gamma(x) = x \) if \( x \neq x_i \). We say that \( \gamma \) has length \( k \) (also denoted by \( |\gamma| \)) and has the set \( (x_1, x_2, \ldots, x_k) \) for domain (or orbit).

Evidently, there are \( (n)_k/k \) cycles of length \( k \) because each cycle \( (x_1, x_2, \ldots, x_k) \) is given by any one of the following \( k \)-arrangements: \( (x_1, x_2, \ldots, x_k), (x_2, x_3, \ldots, x_k, x_1), \ldots, (x_k, x_1, \ldots, x_{k-1}) \), and only by these.

A circular permutation is a cycle of length \( n \) (\( = |N| \)). So there are \( (n)_n/n = (n-1)! \) such permutations. A transposition \( \tau \) is a cycle of length \( 2 \): in other words, there exist two points \( a \) and \( b \), \( a \neq b \), such that \( \tau(a) = b, \tau(b) = a \). There are exactly \( \binom{n}{2} \) transpositions of \( N \).

We recall that each permutation can be written as a product of cycles, with disjoint domains, this decomposition being unique up to order. For example, the permutation of p. 230 can be written as \( (a, c, e, b)(f, g)(d) = (a, c, e, b)(f, g) \) (the cycles of length 1 are often omitted). Similarly, \( e = (x_1)(x_2) \cdots (x_n) \). Currently, the cycles in the sense of graphs (p. 62) and cycles in the sense of permutations will be identified, as in Figure 37. Each cycle is product of transpositions; in fact, \( (x_1)(x_2)(x_3)(x_4) \) and \( (x_1, x_2, \ldots, x_k) = (x_1, x_2)(x_3, x_4) \cdots (x_{k-1}, x_k) \) for \( k \geq 2 \). Hence, this holds for each permutation, because they are products of cycles.

It follows that the set \( \mathfrak{T} = \mathfrak{T}(N) \) of transpositions of \( N \), \( |\mathfrak{T}| = \begin{pmatrix} n \\ 2 \end{pmatrix} \), generates the group \( \mathfrak{S}(N) \). In fact, \( \mathfrak{S}(N) \) can be generated by a much smaller set of transpositions. To make this more precise, let us associate with every set of transpositions \( \mathfrak{U} \subset \mathfrak{T} \) the graph \( g(\mathfrak{U}) \) defined as follows:
Theorem. A set \( U(\subset X) \) of \((n-1)\) transpositions of \( N \) generates \( \Sigma(N) \) if and only if \( g(U) \) is a tree (Definition B, p. 62).

If \( g(U) \) is a tree over \( N \), then for all \( a, b \in N \), \( a \neq b \) there exists a unique path \( x_1 (=a), x_2, \ldots, x_k (=b) \) such that \( \{x_i, x_{i+1}\} \) is an edge of \( g(U) \); hence the transposition \( (x_i, x_{i+1}) \in U \). Now it is easily verified that the transposition \( (a, b) \) can be factored as follows in the group \( \Sigma(N) \):

\[
(a, b) = (x_1, x_a) = (x_{a-1}, x_1) (x_{a-2}, x_{a-1}) \cdots (x_1, x_3) \times (x_2, x_3) \cdots (x_{k-2}, x_{k-1}) (x_{k-1}, x_k).
\]

Thus, as each \( (a, b) \in X \) is generated by \( U \), \( \Sigma(N) \) is too (cf. p. 231).

Now we suppose conversely that \( U \) generates \( \Sigma(N) \), but that \( g(U) \) is not a tree. Because \( g(U) \) has \((n-1)\) edges, there exist \( a \) and \( b \) not connected by a path (Theorem C, p. 63); this implies that the transposition \( (a, b) \) is not equal to any product of transpositions belonging to \( U \), etc. ■

(For other properties related to representing a set of permutations by a graph, see [Dénes, 1959], [Eden, Schützenberger, 1962], [Eden, 1967], and [Berge, 1968], pp. 117-23.)

For two decompositions into a product of transpositions of a given permutation, \( \sigma = \varphi_1 \varphi_2 \cdots \varphi_s \cdots \psi_i \cdots \psi_t \psi_i \cdots \psi_2 \psi_1 \), the numbers \( s \) and \( t \) have the same parity. This can be quickly seen by observing first that the product \( \tau \sigma \) of the transposition \( \tau = (a, b) \) and a permutation \( \sigma \) with \( k \) cycles is a permutation with \( k+1 \) cycles if \( a \) and \( b \) are in the same orbit, and with \( k-1 \) cycles if \( a \) and \( b \) are in different orbits of \( \sigma \). Hence it follows that \( \varphi_1 \varphi_2 \cdots \varphi_s \cdots \psi_i \cdots \psi_t \varphi_1 \cdots \varphi_s \cdots \psi_i \cdots \psi_t \psi_1 \cdots \psi_t \), a number of cycles equal to \( 1 \pm 1 \pm 1 \pm \cdots \pm 1 \), \((s-1)\) times \( \pm 1 \), and \( 1 \pm 1 \pm 1 \pm \cdots \pm 1 \), \((t-1)\) times \( \pm 1 \), respectively. The equality of these two numbers implies the above-mentioned property. (This is the proof by [Cauchy, 1815]. See also [Serret, 1866], II, p. 248.)

A permutation is called even (respectively odd) if it can be decomposed into an even (respectively odd) number of cycles of even length.

The sign \( \chi(\sigma) \) of a permutation \( \sigma \) is defined by \( \chi(\sigma) = +1 \) (respectively -1) if \( \sigma \) is even (respectively odd). From the decomposition into transpositions it follows immediately that for each two permutations \( \sigma \) and \( \sigma' \):

\[
\chi(\sigma \sigma') = \chi(\sigma) \chi(\sigma').
\]

The alternating subgroup of \( \Sigma(N) \) consists of the even permutations of \( N \).

The order of a permutation \( \sigma \) is the smallest integer \( k > 1 \) such that \( \sigma^k = e \). This is clearly the LCM of the system of integers consisting of lengths of the cycles occurring in the decomposition of \( \sigma \).

6.2. Counting problems related to decomposition in cycles; return to Stirling numbers of the first kind

Definition. Let \( c_1, c_2, \ldots, c_n \) be integers \( \geq 0 \) such that:

\[
2c_1 + 2c_2 + \cdots + nc_n = n.
\]

A permutation \( \sigma \in \Sigma(N), |N|=n \) is said to be of type \( [c] = [c_1, c_2, \ldots, c_n] \) if its decomposition into disjoint cycles contains exactly \( c_i \) cycles of length \( i \), \( i=1, 2, 3, \ldots, n \). In other words, the partition of \( N \) given by the orbits of \( \sigma \) is of type \( [c_1, c_2, \ldots] \) (Definition B, p. 205).

Theorem A. A permutation \( \sigma \in \Sigma(N) \) of type \( [c] \) is even (or odd) if and only if \( c_2 + c_4 + c_6 + \cdots \) is even (or odd).

We have already seen this on p. 231. ■

Theorem B. The number of permutations of type \( [c] = [c_1, c_2, \ldots] \) equals:

\[
\nu(n; c_1, c_2, \ldots) = \frac{n!}{c_1! c_2! \cdots c_n! c_1 c_2 \cdots c_n (1^{c_1} 2^{c_2} \cdots n^{c_n})} (0! = 1)
\]

Giving such a permutation of type \( [c] \) is equivalent to giving first a division of \( N \) into the \( c_i \) orbits of length \( i \) of the permutation, with \( i=1, 2, 3, \ldots \); then to erasing for all \( i \) the order on the set of \( c_i \) orbits of length \( i \), and finally to equipping each orbit with a cyclic permutation of its own.
Thus:

\[ p(n; c_1, c_2, \ldots) = \frac{n!}{(1)^{c_1} (2)^{c_2} \cdots (c_1)! c_2! \cdots} \times \frac{1}{(2-1)!} (3-1)! c_2 (3-1)! c_3 \cdots \]

which gives \( [2b] \) after cancellations.

**Theorem C.** Let \( p(n; k; c_1, c_2, \ldots) \) be the number of permutations of \( N \), \( |N| = n \), of type \( [c_1, c_2, \ldots] \), whose total number of orbits (number of cycles in the decomposition) equals \( k \), \( c_1 + c_2 + \cdots = k \). Then we have the following GF in an infinite number of variables \( t, u, x_1, x_2, \ldots \):

\[ \Phi(t, u; x_1, x_2, \ldots) := \sum_{n, k, c_1, c_2, \ldots \geq 0} p(n, k; c_1, c_2, \ldots) \frac{t^n}{n!} x_1^{c_1} x_2^{c_2} \cdots \]

\[ = \exp \left\{ u \left( x_1 t + x_2 \frac{t^2}{2} + x_3 t^3 + \cdots \right) \right\}. \]

In fact, \( p(n, k; c_1, c_2, \ldots) = p(n; c_1, c_2, \ldots) \) if \( c_1 + c_2 + \cdots = k \) and \( c_1 + 2c_2 + \cdots = n \); if not, \( p(n, k; c_1, c_2, \ldots) = 0 \). Hence, by \( [2b] \):

\[ \Phi(t, u; x_1, x_2, \ldots) := \sum_{n, k, c_1, c_2, \ldots \geq 0} \frac{n!}{c_1! c_2! \cdots} \frac{t^{c_1+c_2+\cdots}}{n!} x_1^{c_1} x_2^{c_2} \cdots \]

\[ = \exp \left\{ u \left( x_1 t + x_2 \frac{t^2}{2} + x_3 t^3 + \cdots \right) \right\}. \]

**Theorem D.** The number of permutations of \( N \) with \( k \) orbits (whose decomposition has \( k \) cycles) equals the unsigned Stirling number of the first kind \( s(n, k) \).

The required number, say \( a(n, k) \), equals the sum of the \( p(n, k; c_1, c_2, \ldots) \), taken over all systems of integers \( c_1, c_2, \ldots \) such that \( c_1 + c_2 + \cdots = k \) and \( c_1 + 2c_2 + \cdots = n \). Hence, by \( [2c] \):

\[ \sum_{n, k \geq 0} a(n, k) \frac{t^n}{n!} u^k = \Phi(t, u; 1, 1, 1, \ldots) \]

Hence \( a(n, k) = s(n, k) \) by \([5a, d]\) (p. 212).
By [3a], \( P_n = \binom{n}{2} (2 P_{n-1} + (n-1) P_{n-2}) \).

By [3a], \( P_n = \binom{n}{2} \sum_{i=1}^{n-1} \binom{n}{2} (i) \cdot \binom{n-i}{2} \cdot \binom{n-i}{2} \cdot \binom{n-i}{2} \).

\[ f(t) := \sum_{n \geq 0} \frac{P_n t^n}{n!} = \frac{e^{-t/2}}{\sqrt{1 - t}} \]

\[ P_n = \binom{n}{2} (2 P_{n-1} + (n-1) P_{n-2}) \]

By [3a], \( P_n = \binom{n}{2} \sum_{i=1}^{n-1} \binom{n}{2} (i) \cdot \binom{n-i}{2} \cdot \binom{n-i}{2} \cdot \binom{n-i}{2} \).

\[ (2n-2a)!/2^{-a} = \text{QED} \]

The GF [3c] follows then from the explicit formula [3b]. As for the recurrence relation [3d], this follows from the differential equation \( (1 - t f') = \frac{1}{(1 - t)^2} \)

By Theorem A, one can deduce for \( P(n, k) \) more and more complicated formulas. For instance,

\[ P(n, 3) = \frac{1}{36^n} \sum_{q_1+q_2+q_3=n} (-1)^{q_1} (3^q_1 + 3^q_2) \times \alpha_2! \alpha_3! (\alpha_1, \alpha_2, \alpha_3)^2 2^{q_1} 12^{q_2}, \]

from which one may deduce a linear recurrence relation for \( P(n, 3) \) with coefficients that are polynomials in \( n \).

It is often convenient to represent a permutation \( \sigma \in S[n] \) by a polygon whose sides are segments \( A_i, A_{i+1}, i \in [n-1] \) such that \( A_i \) has \( i \) for 'abscissa' and \( \sigma(i) \) for 'ordinate'. The heavy line in Figure 39 represents the polygon of \( \sigma \in S[7] \), defined by the cycle \( (1, 3, 5, 2) \), in the sense of p. 231; hence, the points 4, 6, 7 are fixed points.

**Definition.** An inversion of a permutation \( \sigma \in S[n] \) is a pair \( (i, j) \) such that \( 1 < i < j < n \) and \( \sigma(i) > \sigma(j) \). In this case we say that \( \sigma \) has an inversion in \( (i, j) \).

Hence, in the associated polygon, an inversion 'is' a segment \( A_iA_j \), \( 1 < i < j < n \), with negative slope. The permutation which is represented in Figure 39 induces 5 inversions, whose corresponding segments are indicated by thin lines.

Let \( I_\sigma \) be the number of inversions of \( \sigma \in S[n] \). Clearly, \( 0 \leq I_\sigma \leq \binom{n}{2} \), with \( I_\sigma = 0 \iff \forall i \in [n], \sigma(i) = i \) and \( I_\sigma = \binom{n}{2} \iff \forall i \in [n], \sigma(i) = n - i + 1 \).

**Theorem A.** The sign \( \chi(\alpha) \) (see p. 233) of a permutation \( \alpha \in S[n] \) equals \((-1)^r\).

- We abbreviate \( q(\alpha) := (-1)^r \) and \( [n]_r := \Psi_r [n] \). Then:

\[ q(\alpha) = \prod_{\{i, j \in [n]: i < j\}} \frac{\alpha(i) - \alpha(j)}{i - j}. \]

Hence, for \( \alpha \) and \( \beta \in S[n] \), we obtain by change of variable \( i' = \beta(i) \),
Moreover, the number of inversions \( I \) of a transposition \( \tau:=(a,b) \), which interchanges \( a \) and \( b \), \( 1 \leq a < b \leq n \), can be read off from the polygon of \( \tau \), and it equals \( 2(b-a)-1 \), hence \( q(\tau)=-1 \). Thus, if we write an arbitrary \( \sigma \in S[n] \) as a product of transpositions, it follows, with (4b) for (*)& and p. 233 for (**), that:

\[
(-1)^k q(\sigma) = q(\tau_1 \tau_2 \ldots \tau_s)
\]

\[= q(\tau_1) q(\tau_2) \ldots q(\tau_s) = (-1)^s q(\sigma). \]

**THEOREM B.** The number \( b(n, k) \) of permutations of \([n]\) with \( k \) inversions satisfies the recurrence relations ([Bourget, 1871]):

\[
b(n, k) = \sum_{0 \leq k-i \leq j \leq k} b(n-1, j)
\]

if \( n \geq 1 \);

\[
b(n, 0) = 1 \quad b(0, k) = 0 \quad \text{if} \quad k \geq 1.
\]

Let \( b(n, k) \) be the set of permutations of \([n]\) that induce \( k \) inversions, \( b(n, k) = |b(n, k)| \), and let \( b_i(n, k) \) be the set of the \( \sigma \in b(n, k) \) such that \( \sigma(1)=i \), \( i \in [n] \). Then we have the division:

\[
b(n, k) = \sum_{1 \leq i \leq n} b_i(n, k).
\]

Let \( f \) be the map of \( b_i(n, k) \) into \( b(n-1, k-i+1) \) defined by:

\[
f(\sigma)(j) = \begin{cases} \sigma(j+1), & \text{if } \sigma(j+1) < i \\ \sigma(j+1) - 1, & \text{if } \sigma(j+1) > i \\ \end{cases}, \quad i \in [n-1].
\]

It is clear that \( f \) is a bijection. Hence, if we use the convention:

\[
b(u, v) = 0, \quad \text{if} \quad v < 0 \quad \text{or if} \quad v > \binom{u}{2},
\]

we get, by passing to the cardinalities in [4d]:

\[
b(n, k) = \sum_{1 \leq i \leq n} |b_i(n, k)| = \sum_{1 \leq i \leq n} b(n-1, k-i+1),
\]

in other words, we just obtain [4c], if we do not use the convention [4f] and if we change the summation variable to \( i := k - j + 1 \).

**THEOREM C.** ([Muir, 1898]). The numbers \( b(n, k) \) have as GF:

\[
\Phi_n(u) := \sum_{0 \leq k \leq \binom{n}{2}} b(n, k) u^k = \prod_{1 \leq i \leq n} \frac{1-u^i}{1-u} = (1+u)(1+u+u^2)\ldots(1+u+u^2+\ldots+u^{n-1}).
\]

Using [4c] for (*)& and putting \( i := k - j + 1 \) for (**), we get:

\[
\Phi_n(u)(**) \sum_{0 \leq k \leq \binom{n}{2}} \{u^k \sum_{i \leq k} b(n-1, j)\}
\]

\[= (\sum_{0 \leq j \leq \binom{n-1}{2}} u^{i-1}) \sum_{0 \leq j \leq \binom{n-1}{2}} b(n-1, j) u^j
\]

\[= (1+u+\ldots+u^{n-1}) \Phi_{n-1}(u),
\]

from which [4h] easily follows.

**THEOREM D.** The numbers \( b(n, k) \) satisfy the following relations:

\[
b(n, k) = b(n, k-1) + b(n-1, k), \quad \text{if} \quad k < n.
\]

\[
\sum_{k=0}^{\binom{n}{2}} b(n, k) = n!.
\]

\[
\sum_{k=0}^{\binom{n}{2}} (-1)^k b(n, k) = 0.
\]
(IV) \[ b(n, k) = b \left( \binom{n}{2} - k \right). \]

(V) \[ \sum_{k=0}^{n} k b(n, k) = \frac{n}{2} \binom{n}{2} n! = \sum L_n \] \[ \text{([Henry, 1881]).} \]

(1) From \([4h]\) follows \((1-u) Q_p = (1-u^n) Q_{n-1}\), where the coefficients of \(u^k\) must be identified. (II) Put \(u=1\) in \([4h]\). (III) Put \(u=-1\) in \([4h]\). (IV) Observe that the polynomial \(Q_n(u)\) is reciprocal. (V) Put \(u=1\) in \(\frac{d}{d \Phi_n} du\). ■

N.B. Find also combinatorial proofs of Theorem D!

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Thus, in Figure 40 the 5 rises [4 falls] of a permutation of [10] are indicated by a heavy [thin] line.

Let \(A_n\) be the number of rises of \(\sigma\), in other words, the number of sides with positive slope of the associated polygon. Clearly, \(0 \leq A_n \leq n-1\), and \(A_n = 0 \Leftrightarrow \forall i \in [n], \sigma(i) = n-i+1\), and \(A_n = n-1 \Leftrightarrow \forall i \in [n], \sigma(i) = i\). Moreover, the number of falls of \(\sigma\) is evidently equal to:

\[ [5a] \quad n - 1 - A_n. \]

**Theorem A.** The number \(a(n, k)\) of permutations of \([n]\) with \(k\) rises satisfies the following recurrence relations:

\[ [5b] \quad a(n, k) = (n-k)a(n-1, k-1) + (k+1)a(n-1, k) \]

for \(n, k \geq 1\), with \(a(n, 0) = 1\) for \(n > 0\), and \(a(0, k) = 0\) for \(k > 0\).

Let \(a(n, k)\) be the set of permutations of \([n]\) that induce \(k\) rises. The number \(a(n, k) = |a(n, k)|\) is also the number of permutations of \([n]\) that induce \(k\) falls, which can be seen by associating with \(\sigma \in S[n]\) the permutation \(i \mapsto (n-i+1)\). Hence:

\[ [5c] \quad a(n, k) = a(n, n-k-1). \]

Now we define the map \(g\) of \(a(n, k)\) into \(S[n-1]\) by:
It is clear that \( \sigma' \in \mathcal{A}(n-1, k) \) in the case of Figures 41a, b, and that \( \sigma' \in \mathcal{A}(n-1, k-1) \) in the case of Figures 41c, d.

Conversely, if \( \sigma' \in \mathcal{A}(n-1, k) \), some reflection shows that \(|g^{-1}(\sigma')| = \) the number of rises of \( \sigma' \) (see Figure 41b) + 1 (see Figure 41a) = \( k+1 \); if \( \sigma' \in \mathcal{A}(n-1, k-1) \) we have, similarly, with \( 5a \) for \( (*) \): \(|g^{-1}(\sigma')| = \) the number of falls of \( \sigma' \) (see Figure 41c) + 1 (see Figure 41d) = \((n-1)-1-(k-1)\) + 1 = \( n-k \). Hence:

\[
|a(n,k)| = \sum_{\sigma' \in \mathcal{A}(n-1,k)} |g^{-1}(\sigma')| + \sum_{\sigma' \in \mathcal{A}(n-1,k-1)} |g^{-1}(\sigma')| = (k+1)|a(n-1,k)| + (n-k)|a(n-1,k-1)|
\]

**Theorem B.** Let \( A(n, k) \) denote the Eulerian number (introduced in \([14t], p. 51\)) then we have:

\[
[5e] \quad A(n, k-1) = A(n, k) \quad (8)
\]

In fact, if we put \( \bar{A}(n, k) = a(n, k-1) \), then the recurrence relation \([5b] \) becomes exactly \([14u] \) (p. 51), where \( A(n, k) \) is replaced by \( \bar{A}(n, k) \), **including the initial conditions.** Hence \( \bar{A}(n, k) = A(n, k) \). Equality \([5e] (8) \) follows then from \([5c] \).

Evidently, \( \sum_{k} A(n, k) = n! \) and, by \([5b] \).

\[
[5e'] \quad A(n, k) = (n-k+1)A(n-1, k-1) + kA(n-1, k).
\]

**Table of Eulerian numbers \( A(n, k) \)**

<table>
<thead>
<tr>
<th>( n )</th>
<th>( 1 )</th>
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<td>15619</td>
<td>4293</td>
<td>247</td>
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</tr>
<tr>
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<td>1</td>
<td>120</td>
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</tbody>
</table>

\[\text{[*David, Kendall, Barton, 1966, p. 260, } n \leq 16.\]

**Theorem C.** The Eulerian numbers \( A(n, k) \) have the value:

\[ [5f] \quad A(n, k) = \sum_{0 \leq j \leq k} (-1)^j \binom{n+1}{j} (k-j)^n. \]

\[\text{Use the GF } [14v] \text{ of p. 51, and equate the coefficients in the first and last member of } [5g] \text{ of } u^k/n!: \]

\[ [5g] \quad 1 + \sum_{i \geq 1} A(n, k) \frac{r^i}{n!} u^k = \frac{1-u}{1-ue^{u(1-u)}} = \]

\[ = (1-u) \sum_{i \geq 0} \frac{u^i}{i!} \sum_{l \geq 0} \frac{u^l}{l!} \sum_{n \geq 0} n^l A(n, k) \frac{u^l}{l!} (1-u)^{n+l} = \]

\[ = \sum_{n \geq 0} \sum_{i \geq 0} (-1)^i \binom{n+i}{i} \binom{n+1}{h} \frac{u^i}{i!} (1-u)^{n+l}. \]

If \( k > n \), then \( A(n, k) = 0 \), and \([5f] \) implies an interesting identity in that case.

**Theorem D.** The Eulerian numbers \( A(n, k) \) satisfy:

\[ [5h] \quad x^n = \sum_{i \geq k \leq n} A(n, k) \binom{x+k-1}{n}. \]

\[\text{[Worpitzky, 1883]. For other properties and generalizations see } [Abram-}
As identity \([5h]\) is polynomial in \(x\), of degree \(n\), it suffices to verify it for \(x = 0, 1, 2, \ldots, n\), which comes down to 'inverting' \([5f]\) in the sense of p. 143. By \([5f]\), for \((\ast)\), we get (cf. Exercise 5 (3), p. 221):

\[
\sum_k A(n, k) i^k = \sum_{0 \leq i \leq k} (-1)^{k-i} \binom{n+1}{k-i} i^k = \sum_{i=0} \left\{ i^k \sum_{k \geq i} \frac{n+1}{k-i} (-i)^{k-i} \right\} = (1-i)^{n+1} \sum_{i=0} i^k.
\]

Hence \(\sum_{i \geq 0} i^t = (1-i)^{n+1} \sum_{k=1}^n A(n, k) i^k\), in other words we have for the coefficient of \(i^k\): 

\(i^k = \sum_k A(n, k) \binom{n+i-k}{n}\), hence \([5h]\) with \(x = i\) and \([5e]\) \((\ast)\). 

We now introduce the Eulerian polynomials \(A_n(u) = \sum A(n, k) u^k\); 

\(A_0(u) = 1, A_1(u) = u, A_2(u) = u + u^2, A_3(u) = u + 4u^3 + u^2, \ldots\)

Taking \([14v]\) p. 51 into account for \([5i]\), and \([14t]\) p. 51 for \([5j]\), we have the following GF:

\[
[5i] \quad \sum_{n \geq 0} A_n(u) \frac{t^n}{n!} = \frac{(1-u)}{1 - u e^{(1-u)}}\]

\[
[5j] \quad \frac{1}{u} \sum_{n \geq 1} A_n(u) \frac{t^n}{n!} = \frac{1-u}{e^{(u-1)}-u}.
\]

\[
[5k] \quad \sum_{n \geq 0} \frac{A_n(u)}{u(u-1)^t} \frac{t^n}{n!} = \frac{1-u}{e^t - u},
\]

the last one, \([5k]\), follows from \([5j]\), where \(t\) is replaced by \(t/(u-1)\).

**Theorem E** (Frobenius). The Eulerian polynomials are equal to:

\[
[5i] \quad A_n(u) = u \sum_{k=1}^n k! S(n, k) (u-1)^{n-k}
\]

\[
[5j] \quad \hat{A}(t, u) = 1 + \sum_{n \geq 0} A(n, k) \frac{t^n}{n!} u^{k-1} = \frac{1-u}{e^{(u-1)}-u}.
\]

\[
[5k] \quad \hat{A}(t, u) = \sum_{0 \leq k \leq n} A(n, k) \frac{t^n}{n!} u^k = \frac{1-u}{1 - u e^{(1-u)}}.
\]

The above-mentioned GF of the Eulerian numbers, namely

\[
[5o] \quad \hat{A}(t, u) = 1 + \sum_{1 \leq k \leq n} A(n, k) \frac{t^n}{n!} u^{k-1} = \frac{1-u}{e^{(u-1)}-u}.
\]

have the disadvantage of being asymmetric. Everything becomes easier if we introduce the symmetric Eulerian numbers \(\hat{A}(l, m)\) defined by:

\[
[5q] \quad \hat{A}(l, m) = A(l + m + 1, m + 1).
\]
The table of these is obtained from the table on p. 243 by sliding all columns upward:

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<tr>
<th>l \ m</th>
<th>0</th>
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<td>26</td>
<td>302</td>
<td>2416</td>
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</tbody>
</table>

**Theorem G ([Carlitz, 1969]).** We have the following GF:

\[
\sum_{l,m \geq 0} A(l,m) \frac{x^l y^m}{(l+m+1)!} = \frac{e^x - e^y}{1 - xy}.
\]

In fact, by [5p] for (*), the left-hand member of [5r] equals:

\[
\sum_{m \geq 0} A(l+m+1,m+1) x^l y^m = (1/y) \sum_{m \geq 0, n \geq m+1} A(n,m+1) \times (y/x)^{n+1} x^n y^{n!} (1/y) (-1 + \frac{1}{y}, y/x),
\]

providing the second member of [5r] after simplifications.

The following is a generalization of the problem of the rises, often called the 'problem of Simon Newcomb'. Instead of permuting the set [\[n\]], one permutes a set \(P, |P|=p\), consisting of \(c_1\) numbers 1, \(c_2\) numbers 2, ..., \(c_n\) numbers \(n\), \(c_1+c_2+\cdots+c_n=p\), and we want to find the number of permutations with \(k-1\) rises. ([Kreweras, 1965, 1966b, 1967], [Riordan, 1958, p. 216; cf. Exercise 21, p. 265.) In more concrete terms, one draws from a set of 52 playing cards all cards, one by one, stacking them on piles in such a way that one starts a new pile each time a card appears that is 'higher' than its predecessor. In how many ways can one obtain \(k-1\) piles? (here \(c_1=c_2=\cdots=c_{13}=4\).

6.6. Groups of permutations; cycle indicator polynomial; Burnside theorem

**Definition A.** A group \(\mathfrak{G}\) of permutations of a finite set \(N\) is a subgroup of the group \(\mathfrak{S}(N)\) of all permutations of \(N\). We denote \(\mathfrak{G} \subseteq \mathfrak{S}(N)\). \(|\mathfrak{G}|\) is called the order of \(\mathfrak{G}\), and \(|N|\) its degree.

Thus, the alternating group is a permutation group of \(N\), of order \(n!/2\).

For each permutation \(\sigma \in \mathfrak{S}(N), N=\{A, B, C, D, E, F\}\) for which we have,
by \([6a]\): \(c_1(\sigma) = 2, c_2(\sigma) = c_3(\sigma) = 0, c_4(\sigma) = 1, c_5(\sigma) = c_6(\sigma) = 0\), hence the monomial \(\frac{1}{x^2} x_1^2 x_4^2\) in \(Z(x)\). There are 6 kinds of rotations, which can be described by Figures 42a, b, c, namely, a rotation of \(\pi/2\) or \(\pi\) or \(3\pi/2\) around a line joining the centers of opposite faces (Figure 42a), a rotation of \(\pi\) around a line joining the centers of opposite edges (Figure 42b) and rotations of \(2\pi/3\) or \(4\pi/3\) around a line joining opposite vertices (Figure 42c). Making up the list of permutations of each kind, we finally find, by \([6c]\):

\[
Z(x) = \frac{1}{x^2} (x_1^6 + 3x_1^2 x_2^2 + 6x_1^2 x_4 + 6x_2^3 + 8x_3^3).
\]

**Definition C.** The stabilizer of \(x\) (\(\in N\)) with respect to \(\mathcal{G}(\leq\mathcal{S}(N))\), denoted by \(\mathcal{G}(x)\), is the set of permutations \(\sigma \in \mathcal{G}\) for which \(\sigma(x) = x\).

It is clear that \(\mathcal{G}(x)\) is a subgroup of \(\mathcal{G}\).

**Definition D.** For \(\mathcal{G} \leq\mathcal{S}(N)\), the orbit of \(x (\in N)\) under \(\mathcal{G}\), denoted by \(x^\mathcal{G}\), is the set of all \(y \in N\) for which there exists \(\sigma \in \mathcal{G}\) such that \(y = \sigma(x)\).

In particular, the orbit of \(x\) under the subgroup \((\sigma)\) generated by \(\sigma\), \(\sigma = \{e, \sigma, \sigma^2, \ldots\}\) is just \(x^{(\sigma)} = \{x, \sigma(x), \sigma^2(x), \ldots\}\) (see p. 231). For \(x \neq x'\) either \(x^\mathcal{G} = x'^\mathcal{G}\) or \(x^\mathcal{G} \cap x'^\mathcal{G} = \emptyset\). The set \(\Omega\) or all (different) orbits is hence a partition of \(N\), \(N = \sum_{\omega \in \Omega} \omega\).

**Theorem A (on the stabilizer).** For every \(x \in N\) and every group \(\mathcal{G} \leq\mathcal{S}\), the order of \(\mathcal{G}\) equals the product of the order of the stabilizer \(\mathcal{G}(x)\) by the size of the orbit \(x^\mathcal{G}\):

\[
[6a] \quad |\mathcal{G}(x)| \cdot |x^\mathcal{G}| = |\mathcal{G}|.
\]

In other words, denoting by \(\Omega\) the set of orbits, \(\sum_{\omega \in \Omega} \omega = N\):

\[
[6h] \quad \forall \omega \in \Omega \Rightarrow |\mathcal{G}(x)| \cdot |\omega| = |\mathcal{G}|.
\]

It is clear that for each permutation \(\sigma \in \mathcal{G}\):

\[
|\mathcal{G}(x)| \cdot |\sigma(\mathcal{G}(x))| = |\mathcal{G}(x)|,
\]

where \(\sigma(\mathcal{G}(x)) = \{x \beta \beta(\mathcal{G}(x)) \in \mathcal{G}(x)\}\) is a left coset of the subgroup \(\mathcal{G}(x)\) of \(\mathcal{G}\).

For each \(y\) of the orbit of \(x\), \(y \in x^\mathcal{G} = \omega\) (Figure 43) we choose one single permutation \(\sigma = \sigma y \in \mathcal{G}\) such that \(y = \sigma(x)\), and we consider the map \(f: y \mapsto \sigma(y)\). It is easily verified that \(f\) is a bijection of \(\mathcal{G}\) into the set of left cosets of \(\mathcal{G}(x)\). All these cosets have the same number of elements, \([6i]\), and since they constitute together a partition of \(\mathcal{G}(x)\), we get: \(|\mathcal{G}| = \text{the number of elements in every class } \times \text{ the number of classes}= |\mathcal{G}(x)| \cdot |\mathcal{G}|\).

**Theorem B (Burnside-Frobenius).** Let \(\Omega\) stand for the set of orbits of \(\mathcal{G}\). Then we have:

\[
[6j] \quad |\Omega| = \frac{1}{|\mathcal{G}|} \sum_{\sigma \in \mathcal{G}} |N_0(\sigma)|,
\]

where \(N_0(\sigma)\) is the set of fixed points of \(\sigma\).

Let \(E\) be the set of pairs \((x, \sigma), \sigma \in \mathcal{G}\) such that \(\sigma(x) = x\). Clearly, we have the following divisions:

\[
[6k] \quad E = \sum_{\sigma \in \mathcal{G}} \{(x, \sigma) \mid \sigma(x) = x\} - \sum_{x \in N} \{(x, \sigma) \mid \sigma(x) = x\}.
\]
Now, for fixed \( \sigma \), \( |\{(x, \sigma, \sigma(x)=x)\}|=|\{x \in \mathbb{N}, \sigma(x)=x\}|=|N_0(\sigma)| \) and for fixed \( x \), \( |\{(x, \sigma) \sigma(x)=x\}|=|\{\sigma \in \mathcal{G}, \sigma(x)=x\}|=|\mathcal{G}(x)| \). Hence, by passing to the cardinals in [6k], and with \([6h]\) for \((*)\):

\[
|E| = \sum_{\sigma \in \mathcal{G}} |N_0(\sigma)| = \sum_{x \in \mathbb{N}} |\mathcal{G}(x)| = \sum_{\sigma \in \mathcal{G}} \left( \sum_{x \in \mathbb{N}} |\mathcal{G}(x)| \right)
\]

\[
= \sum_{\sigma \in \mathcal{G}} \sum_{x \in \mathbb{N}} |\mathcal{G}(x)| - \sum_{\sigma \in \mathcal{G}} |\mathcal{G}| = |\mathcal{G}|^2
\]

6.7. Theorem of Pólya

(I) An example

In order to clarify the aim of this section consider the following problem. In how many ways can one paint the six faces on a cube in at most \( c \) colours, it being understood that two colourings will not be distinguished if they can be transformed into each other by a rotation of the cube. In the last case the colourings are called equivalent. The class of colourings that are all equivalent to a given one, is called a model or a configuration. For example, in the case of two colours, white and blue, \( c=2 \), the colourings \{blue: E, F, white: the rest\} and \{blue: A, C\} are equivalent (Figure 44), but these two are not equivalent to the colouring \{blue: A, B\}. Direct counting shows that there are only 10 models for all possible \( 2^6 = 64 \) possible colourings. Figure 45 shows the 6 models corresponding with at most 3 blue faces (blue=hatched), the 4 remaining models can be obtained from the set of models with at most 2 blue faces, by interchanging the colours white and blue.

(II) Statement of the problem

Let \( D \) and \( R \) be two finite sets, \(|D|=d, |R|=r\), and let \( \mathcal{G} \) be a group of permutations of \( D \). \( F=R^D \) is the set of maps of \( D \) into \( R \), and \( \mathcal{G} \) is the partition of \( F \) consisting of the \( \sim \) equivalence classes on \( F \) defined by:

\[
[7a] \quad f \sim g \iff \exists \sigma \in \mathcal{G}, \quad g = f(\sigma),
\]

which means: \( \forall x \in D, \quad g(x) = f(\sigma(x)) \).

This is an equivalence indeed, because (I) \( f=f(\varepsilon) \), (II) \( g=f(\sigma) \Rightarrow f=g(\sigma^{-1}) \), (III) \( g=f(\sigma), \quad h=g(\beta) \Rightarrow h=g(\alpha\beta) \). Each class \( f \in \mathcal{G} \) is called a model.

Let also \( A \) be a commutative ring, and \( w \) a map from \( R \) into \( A \), called weight.

We define the weight of \( f \in F \) by:

\[
[7b] \quad W(f) := \prod_{x \in D} w(f(x)),
\]

and the inventory of each subset \( F' \subseteq F \), denoted by \( W(F') \), by:

\[
[7c] \quad W(F') := \sum_{f \in F'} W(f).
\]

It is easy to see, by \([7a, b]\), that:

\[
[7d] \quad f \sim g \Rightarrow W(f) = W(g);
\]

thus we can define the weight \( \mathfrak{W}(f) \) of a model \( f \in \mathcal{G} \) by:

\[
[7e] \quad \mathfrak{W}(f) := W(f), \quad \text{where} \quad f \in \mathcal{G}.
\]
Let \( \mathcal{M}(\mathcal{Y}) \) be the set of the faces of the cube, \( R \) is the set with two elements, 'blue' and 'white'. The weight function \( w \) is defined by \( w(\text{blue}) = t, w(\text{white}) = u \). \( \mathcal{A} \) is the ring of polynomials in two variables \( t, u \). \( \mathcal{G} \) is the group of permutations of the faces of the cube, which we studied already on p. 248; \( \mathcal{F} \) is the set of colourings of the fixed cube, and \( \mathcal{S} \) is the set of models of colourings. If \( W(f) = t^p u^q \), this means, by [7b], that the colouring \( f \) is of type \((p, q)\) in the sense that \( f \) contains \( p \) blue faces and \( q \) white faces, \( p + q = 6 \). Hence, we have:

\[
\mathcal{M}(\mathcal{Y}) = \sum_{f \in \mathcal{F}} \mathcal{M}(f) = \sum_{p,q} \nu(p,q) t^p u^q = P(t, u),
\]

where \( \nu(p,q) \) is the number of models of type \((p,q)\). The total number of models is then equal to:

\[
\sum_{p,q} \nu(p,q) = P(1,1).
\]
as defined on p. 252, we have \( \sum_{\nu \in R} \nu^k(y) = t^k + u^k \); hence by [6f] (p. 248) and [7g, i]:

\[
\begin{align*}
\text{[7o]} \quad P(t, u) &= \frac{1}{2!} \{(t + u)^6 + 3(t + u)^3(t^2 + u^2)^2 + 6(t + u)^2 \times (t^4 + u^4) + 6(t^2 + u^2)^3 + 8(t^2 + u^2)^2 \} \\
&= t^6 + t^5u + 2t^4u^2 + 2t^3u^3 + 2t^2u^4 + tu^5 + u^6.
\end{align*}
\]

For instance, by [7g], the number of colourings with 4 blue faces and two white faces is equal to the coefficient of \( t^4u^2 \) in \([70]\), hence 2. More generally, if there are \( c \) colours, then we have in \([7i]\) \( \nu^c(y) = t_1 + t_2 + \cdots + t_c \), where \( t_1, t_2, \ldots, t_c \) are \( c \) variables. Hence, by notation of Exercise 9 (p. 158), for the monomial symmetric functions:

\[
\begin{align*}
\text{[7p]} \quad P(t_1, t_2, \ldots, t_c) &= \frac{1}{c!} \{(t_1 + t_2 + \cdots + t_c)^6 + \\
&+ 3(t_1 + t_2 + \cdots + t_c)^3(t_1^2 + t_2^2 + \cdots + t_c^2)^2 + \cdots \} = \\
&= \sum \binom{c}{1} t_1^6 + \sum \binom{c}{2} t_1^3 t_2^3 + 2 \sum \binom{c}{3} t_1^3 t_2^3 t_3^3 + \cdots \\
&+ \sum \binom{c}{c} t_1^c + 3 \sum \binom{c}{c} t_1^c t_2^c + 5 \sum \binom{c}{c} t_1 t_2 t_3 t_4^2 + 6 \sum \binom{c}{c} t_1^2 t_2^2 t_3^2 t_4^2.
\end{align*}
\]

For instance, there are 15 models of the cube that use 5 given colours for the faces (hence one colour is used twice). The total number \( v_c \) of models of cubes with at most \( c \) colours is obtained by putting \( t_1 = t_2 = \cdots = t_c = 1 \) in \([7p]\). Then we obtain, after simplifications:

\[
\begin{align*}
v_c &= c + 8 \left( \binom{c}{2} + 30 \binom{c}{3} + 62 \binom{c}{4} + 75 \binom{c}{5} + 30 \binom{c}{6} \right) \\
v_2 &= 10, \quad v_3 = 57, \quad v_4 = 234, \text{ etc.}
\end{align*}
\]


**SUPPLEMENT AND EXERCISES**

1. **Cauchy identity.** Show that \( \sum \{c_1c_2! \cdots \cdots c_{n-1}!2^{n-1} \cdots \cdots 2 \}^{-1} = 1 \), where the summation is taken over all sequences of integers \( c_i \geq 0 \) such that \( c_1 + 2c_2 + \cdots = n \).

2. Return to the permutations with a given number of inversions. Determine an explicit formula of minimal rank for the number \( b(n, k) \) of permutations \([n] \) with \( k \) inversions (cf. p. 237): \( b(n, 1) = n! - 1, b(n, 2) = \binom{n}{2} - 1 \), \( b(n, 3) = \binom{1}{3} n(n^2 - 7), \) \( b(n, 4) = \binom{1}{4} n(n + 1)(n^2 + n - 14), \) \( \ldots \). [Hint: [4h], p. 239, and the 'pentagonal' theorem of Euler, [5g], p. 104.]

3. \( \mathcal{S}[n] \) and \( \mathcal{G}(N) \) as metric spaces. (1) The expression \( d(\sigma, \beta) = \max_{1 \leq i < j \leq n} |x(i) - \beta(j)| \), where \( \sigma \) and \( \beta \) are permutations of \([n] \), defines a distance on the set \( \mathcal{S}[n] \) of all permutations of \([n] \). Let \( \Phi(n, r) \) be the number of elements of an arbitrary ball of radius \( r \), in other words, the number of permutations \( \sigma \) such that \( d(\sigma, e) \leq r \), where \( e \) stands for the identity permutation. Then, \( \Phi(n, 1) = f_n \), the Fibonacci number (p. 45). Moreover, \( \Phi(n, 2) = 2\Phi(n-1, 2) + 2\Phi(n-3, 2) = \Phi(n-5, 2) \) ([Lagrange (R.), 1962a], [Mendelsohn, 1961]). More generally, the computation of \( \Phi(n, r) \) is essentially the computation of a permanent (Exercise 13, p. 201). Between two elements of \( \alpha, \beta \) one can define also another distance function, namely the number of inversions of \( \sigma \beta^{-1} \). (2) For each permutation \( \sigma \in \mathcal{S}[n], N \) finite, let \( N(\sigma) \) be the set of the mobile points of \( \sigma \). Show that \( d(\sigma, \beta) = |N(\sigma \beta^{-1})| \) defines a distance on \( \mathcal{G}(N) \). How many points are there in the ball \( \{ \sigma \ | \ d(\sigma, e) \leq k \} \)? Cf. p. 180.

4. Labeling \( \mathcal{S}[n] \) by inversions. For every permutation \( \sigma \in \mathcal{S}[n] \) and every integer \( k \in [n] \), let \( x_k = x_k(\sigma) \) be the number of integers \( j < k \) such that the pair \((i, k+1)\) is an inversion \((\sigma(j) > \sigma(k+1))\). Evidently \( x_k \leq k \). So we can associate with \( \sigma \) the integer \( x = x(\sigma) = x_1 + 2!x_2 + 3!x_3 + \cdots + (n-1)!x_{n-1} \leq n! - 1 \). Conversely, using the factorial representation of integers (Exercise 9, p. 117), show that each \( x, 0 \leq x \leq n! - 1 \) is the label of a single permutation \( \sigma \); how to determine this permutation? [Example: \( 1 \ 2 \ 3 \ 4 \ 5 \ 6 \) has for label \( 1!1 + 1!3 + 4!4 + 4!5 = 583 \).]

5. \( \mathcal{G}(N) \) as a lattice. We associate with every permutation \( \sigma \in \mathcal{G}(N) \) the subset \( E(\sigma) \subseteq \mathcal{S}[n] \) consisting of the pairs \([i, j]\) which are not inverted: \( i < j \Rightarrow \sigma(i) < \sigma(j) \). Show that \( \sigma \leq \sigma' \) if \( E(\sigma) \subseteq E(\sigma') \) endows \( \mathcal{S}[n] \) with a lattice structure ([Guilbaud, Rosenstiehl, 1960]).
6. **Conditional permutations.** Let \( a \) be a sequence of integers \( 1 \leq a_1 < a_2 < a_3 < \cdots \) and let \( \zeta(n, k; a_1, a_2, \ldots) \) be the number of permutations of \( N, |N| = n \), with \( k \) orbits, such that each has a number of elements equal to one of the \( a_i \). Then ([Gruder, 1953]):

\[
\sum_{n,k} \zeta(n, k; a_1, a_2, \ldots) \frac{i^n}{n!} u^k = \exp \left\{ \left( \frac{t^{a_1}}{a_1} + \frac{t^{a_2}}{a_2} + \cdots \right) \right\}.
\]

More generally, prove a theorem analogous to Theorem B (p. 98) for permutations.

7. **Derangements by number of orbits.** Let \( d(n, k) \) be the number of derangements of \( N, |N| = n \), with \( k \) orbits (p. 231), or permutations with \( k \) cycles of length \( \geq 2 \). (1) We have the following GF: \( e^{-x}(1-x)^{n-1} = 1 + \sum_{k \geq 2} d(n, k) \frac{t^k}{k!} \). [Hint: Use [2b], p. 233.] Hence,

\[
\sum_{k \geq 2} (-1)^{k-1} d(n, k) = n-1.
\]

(2) The following recurrence relation holds: \( d(n+1, k) = n \{d(n, k) + d(n-1, k-1)\} \), \( d(0, 0) = 1 \). ([Appell, 1880], [Carlitz, 1958], [Trikomi, 1951], and Exercises 11 (p. 293) and 20 (p. 295) about the associated Stirling numbers of the first kind, \( s(n, k) = \frac{(-1)^{n-k} n!}{k!} \).

(3) For \( k \geq 2 \), and \( p \) prime, we have \( d(p, k) \equiv 0 \) (mod \( p \)). (4) For all integers \( l \), \( \sum_{m=1}^{\infty} \frac{(-1)^m}{m} d(l+m, m) = (-1)^l l! \). (5) Similarly, \( \sum_{m=1}^{\infty} \frac{(-1)^m}{m} d(l+m, l) = 0 \). (6) We have \( d(2k, k) = (2k-1)(2k-3) \cdots 3 \cdot 1 \), \( d(2k+1, k) = 2k(2k-1) \cdots 3 \cdot 1 \), \( d(2k+2, k) = (4k+2)!/(k+1)! \). A table of the \( d(n, k) \) is given now:

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8. The \( d(n, k) \) above are used in the asymptotic expansion of \( Z_n(t) = t^n + 2t^{n-1} + \cdots + n!t + \cdots \). Let \( [r, q] := e^{-q}(1-e^{-q})^{-r-1}, r \in \mathbb{C}, \text{Re}a > 0 \). Then:

\[
Z_n(t) \approx q^n \sum_{k \geq 0} C_k t^k,
\]

where \( C_k = \sum d(q, q-k) A(q, r) (-a)^{q-k}[r, q]/q! \), a double finite summation where \( k \leq q \leq 2k, r \leq q \), and where the \( A(q, r) \) are the Eulerian numbers of the first kind.

9. The number of solutions of \( \sigma^m = \varepsilon \) in \( \mathfrak{S}(N) \). Let \( T_n \) be the number of permutations \( \sigma \in \mathfrak{S}(N), |N| = n \), such that \( \sigma^2 = \varepsilon \) (= the identity permutation). Such a permutation, or involution (or selfconjugate permutation of Muir) has a cycle decomposition consisting of transpositions only. Deduce the following relations: \( T_n = T_{n-1} + (n-1) T_{n-2} \), \( T_0 = 1 \), and \( \sqrt{n} \leq t \leq \sqrt{n+1} \). Finally, \( \sum_{n \geq 0} T_n t/n! = \exp(t + t^2/2) \). Show that then \( T_n = n! \sum (l \mid 2l) \exp(l + l^2) \) where the summation takes place over the pairs (l, j) such that \( i + j = n \). More generally, let \( T(n, k) \) be the number of solutions of \( \sigma^m = \varepsilon \), \( \sigma \in \mathfrak{S}(N) \) (hence \( T_n = T(n, 2) \)); show that \( \sum_{n \geq 0} T(n, k) t^n/n! = \exp(\sum_{l \mid k} t^l/d) \), where the last summation is taken over all divisors \( d \) of \( k \). (See [Chowla, Herstein, Moore, 1952], [Chowla, Herstein, Scott, 1952], [Jacobstahl, 1949], [Moser, Wyman, 1955a], [Nicolas, 1969].)

Use this to obtain the recurrence relation \( T(n+1, k) = \sum_{d \mid k}{d(n-1, d-1) \times T(n-d+1, k)} \) and the first values of \( T(n, k) \):
10. Permutations with ordered orbits; outstanding elements ([Sade, 1955]).
For each subset $A \subseteq [n]$, we denote by $i(A)$ the smallest integer $\in A$, called the initial integer of $A$. Let $\sigma$ be a permutation of $[n]$, $\sigma \in \Sigma[n]$, whose orbits are numbered, say $\Omega_1(\sigma), \Omega_2(\sigma), \ldots, \Omega_{l}(\sigma)$, such that $i(\Omega_1(\sigma)) < i(\Omega_2(\sigma)) < \cdots < i(\Omega_{l}(\sigma))$, $1 \leq l \leq n$. (1) Let $F(n, k)$ be the number of $\sigma \in \Sigma[n]$ such that $n \in \Omega_l(\sigma)$. Show that

$$F(n, k) = (n - 2)F(n - 1, k) + F(n - 1, k - 1), F(n, 1) = n!, F(n, n) = 1.$$ 
Make a complete study of this double sequence $F(n, k)$. (Find its GF, establish recurrence relations, etc.)

(2) Let $g(n, k, c)$ be the number of permutations of $[n]$ whose $k$-th orbit has $c$ elements. Then $g(n, k, c) = (n - 1)g(n - 1, k, c) + g(n - 1, k - 1, c)$.

(3) An outstanding element $j(\in [n])$ (of $\sigma \in \Sigma(n)$) is, by definition, an element such that $\sigma(j) > \sigma(i)$ for all $i < j$. We make the convention of calling 1 outstanding too. Show that the number of permutations of $[n]$ with $k$ outstanding elements equals $s(n, k)$ ([Rényi, 1962]).

11. Alternating permutations of André, Euler numbers and tangent numbers.
(For an exhaustive study of this problem, see [André, 1879a, 1881, 1883a, 1894, 1895], and [Entringer, 1966] for a reformulation. The expressions we find for $\cos(t)^{-1}$ and $\tan(t)$ give a combinatorial interpretation of the Euler and Bernoulli numbers, [[14a, b], p. 48, and Exercise 36, p. 88.]

We will call a permutation $\sigma \in \Sigma[n]$ alternating if and only if the $(n - 1)$ differences $\sigma(2) - \sigma(1)$, $\sigma(3) - \sigma(2)$, ..., $\sigma(n) - \sigma(n - 1)$ have alternating signs. For example, $(1 \ 2 \ 3 \ 4)$ and $(1 \ 2 \ 3 \ 4)$ are alternating, but $(1 \ 2 \ 3 \ 4)$ and $(1 \ 2 \ 3 \ 4)$ are not. We put $A_0 = A_1 = A_2 = 1$ and we let $2A_n$ be the number of alternating permutations of $[n]$, $n \geq 3$. Show that $2A_{n+1} = - \sum_{k=0}^{n} \binom{n}{k} A_k A_{n-k}$ and that $\sum_{n \geq 0} A_n t^n/n! = t g(t/4 + t/2)$. Use this to

obtain:

$$\sum_{n \geq 0} A_{2n} t^{2n}/(2n)! = (\cos t)^{-1}$$

and

$$\sum_{n \geq 0} A_{2n+1} t^{2n+1}/(2n+1)! = \tan t.$$ 

Hence $A_{2n} = E_{2n}$, where $E_{2n}$ is the Euler number (p. 48), and the $A_{2n+1}$, often called tangent numbers, have the following first values ([Knuth, Buckholtz, 1967], for $m \leq 120$; see also [Esterline, 1902], [Schlömilch, 1857], [Schwatt, 1931], [Toscano, 1936]):

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With Exercise 36, p. 88, and p. 49, $A_{2n-1} = (-1)^{n+1} D_{2n} 4^n(2^n-1)/2n = 4^n-1|G_{2n}|$. Also prove the following explicit values:

$$A_{2n} = \sum_{j \leq \frac{n+1}{2}} (-1)^{j+1} \binom{2k}{k-j} j^{2n-2j-1},$$

$$A_{2n+1} = \sum_{j \leq \frac{n}{2}} (-1)^{j+1} \binom{2k}{k-j} (k+1)^{2n-2j-1}.$$ 
Moreover, as a function of the Eulerian polynomials $A_n(u)$ of p. 244, the tangent number $A_{2n+1}$ equals $A_{2n+1}(-1)$. Finally, it may be valuable to introduce other tangent numbers $T(n, k)$ such that $(tg t)/k! = \sum_{n \geq 1} T(n, k) t^n/n!$, in order to compute the $A_{2n+1} = T(2n+1, 1)$. In fact, we have $T(n+1, k) = T(n, k-1) + k(k+1) x n T(k+1, k+1)$, hence the first values of $T(n, k)$:

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Find a formula of rank 2 for $T(n, k)$. Of course, these numbers are inverses (p. 143) of the arctangent numbers $t(n, k)$ defined by \( (\arctan x)^k/k! = \sum_{n=0}^{\infty} t(n, k) x^n/n! \), for which holds $t(n+1, k) = t(n, k-1) - n(n-1)x \times t(n-1, k)$, the first values being:

<table>
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*12. The number of terms of a symmetric determinant. (1) Let be given two permutations \( \alpha, \beta \in \varepsilon(n), |N|=n \). Show that the following relation is an equivalence relation: "if \( \gamma \) is a cycle of \( \alpha \) (or \( \beta \)), then \( \gamma \) or \( \gamma^{-1} \) is a cycle of \( \beta \) (or \( \alpha^{-1} \)). (2) The number of equivalence classes of type \( [\alpha, \beta, \ldots, \gamma] \) equals \( n! \prod \left\{ 1 + \frac{1}{t_i} \left( \frac{1}{2} \right)^{t_i} \right\} \). (3) The total number \( a_n \) of classes satisfies \( \sum_{n=0}^{\infty} a_n x^n/n! = (1-t)^{1/2} \exp(t/2-t^2/4) \). (4) Differences between the numbers of 'even' classes and 'odd' classes, denoted by \( a'_n \), satisfies \( \sum_{n=0}^{\infty} a'_n x^n/n! = (1+t)^{1/2} \exp(t/2-t^2/4) \). (Cf. [3g], p. 277.) (5) It follows that \( a_{n+1} = (n+1) a_n - \binom{n}{2} a_{n-2} \), \( a'_{n+1} = (n-1) a'_{n-1} \times a'_{n-2} \). (6) Show that the numbers \( p_n \) and \( q_n \) of 'positive' and 'negative' terms satisfy \( p_n + q_n = a_n, p_n - q_n = a'_n \). (7) Treat all the preceding questions for the case of 'derangements', in which case the determinant of (6) is supposed to have only 0 on the main diagonal. ([*Pólya, Szegő, II, 1926], p. 110, Exercises 45-46.)

*13. Permutations by number of 'sequences'. (For many other properties, see [André, 1889].) Let \( \sigma \) be a permutation of \([n]\), \( \sigma \in \varepsilon([n]) \). A sequence of length \( l \geq 2 \) of \( \sigma \) is a maximal interval of integers \([i, i+l-1] \) = \( [i, i+1, \ldots, i+l-1] \) on which \( \sigma \) is monotonic. The sequence is called intermediary or left or right according to whether \( 1 \leq i, i+l-1 < n \) or \( i = 1 \) or \( i + l - 1 = n \). A peak of \( \sigma \) is a maximum with respect to \( \sigma \). The peak (in \( i \)) is called intermediary or left or right, when \( 1 < i < n \) or \( \sigma(i-1) < \sigma(i) > \sigma(i+1) \) or \( i = 1 \), \( \sigma(1) > \sigma(2) \) or \( i = n, \sigma(n-1) < \sigma(n) \), respectively. Let \( P(n, s) \) be the set of permutations of \([n]\) with \( s \) sequences, and let \( P_{n,s} = |P(n, s)| \). Using the map \( g_i \) introduced in [3d] (p. 242) from \( P(n, s) \) into \( P(n-1, s)+P(n-1, s-1)+P(n-1, s-2) \), as well as the notations given above, show that \( P_{n,s} = sP_{n-1,s} + 2P_{n-1,s-1} + (n-s)P_{n-1,s-2} \). For all \( n \geq 2k+4 \), \( 1^kP_{n+1} + 3^kP_{n+3} + 5^kP_{n+5} + \ldots = 2^kP_{n,k} + 4^kP_{n+4} + \ldots \). Finally, \( \sum_{n=0}^{\infty} P_{n,s} x^n/n! = (1+u)^{1/2} \left( (1-u)(1-\sin(u+\cos u))-1 \right) \), where \( u = \sin v \).
associate the division \([n] = I_1 + I_2 + \cdots + I_k\), where the components \(I_h\) of \(n\) are the smallest intervals such that \(\sigma(I_h) = n - h + 1\), \(h = 1, 2, \ldots, k\). For example, \(\sigma = 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6\) has the three components \(\{1, 2, 3\}, \{4\}, \{5, 6\}\), and the identity \(e\) has \(n\) components. A permutation is said to be indecomposable if it has one component; so is \(\sigma\) if \(\sigma(k) = n - k + 1\). We denote by \(C(n, k)\) the number of permutations with \(k\) components. Introducing the Euler formal series \(\varphi(t) := \sum_{n \geq 1} C(n, 1) t^n = \frac{1}{(1 - t)^\infty}\) (see also Exercise 34, p. 171), we have the GF:
\[
\frac{1}{\varphi(t)} = (1 + t - 1)^{\infty}
\]
Find a simple recurrence for \(C(n, k)\). [Hint: Use \(t^{d+1} = (1 - t)^{\infty} \varphi(t^{-1})\).]

Here are the first values of \(C(n, k)\):

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**15. Cayley representation of a finite group.** Let \(G\) be a finite multiplicatively written group, \(n = |G|\). With every \(a \in G\) we associate the permutation \(\sigma_a\) of \(G\) defined by \(\sigma_a(x) = ax, x \in G\). Let \(G\) be the group of these permutations, called the Cayley representation of the group \(G\). Show that \(G\) is isomorphic to \(N\) \(\leftrightarrow \sigma_a, \sigma_b = \sigma_{ab}\) and that \(Z(G; x_1, x_2, \ldots) = \sum_{d \mid n} \varphi(d) (x_d)^{n/d}\), where \(d\) runs through the set of divisors \(\geq 1\) of \(n\), and \(\varphi(d)\) is the number of elements \(a \in G\) with order \(d\).

**16. Cube and octahedron.** (1) Let \(N\) be the set of the 8 vertices of a cube, and \(G\) be the group of permutations of \(N\) induced by the rotations of this cube. Then the cycle indicator polynomial \(Z(x)\) equals \(\frac{1}{24} (x_1^8 + 9 x_2^4 + 6 x_4^2 + 8 x_2^2 x_1^6)\). Prove that if \(N\) is the set of the 12 edges, we have \(Z(x) = \frac{1}{24} (x_1^8 + 3 x_2^4 + 6 x_4^2 + 6 x_2^2 x_1^6 + 8 x_4^2)\). (2) Show that there are only three different ways to distribute three red balls, two black balls and one white ball over the vertices of a regular octahedron in euclidean three-dimensional space. The octahedron is supposed to be freely movable. Generalize to \(c\) colours, as on p. 254.

**17. Colourings of a roulette.** (1) Let \(G\) be the cyclic group of order \(n\). Show that \(Z(G; x_1, x_2, \ldots) = \sum_{d \mid n} \varphi(d) (x_d)^{n/d}\), where \(\varphi(d)\) is the Euler function (p. 193), and \(d \mid n\) means \('d\ divides \(n\').

(2) Now consider a roulette. This is a disc freely rotating around its axis, and divided into \(n\) equal sectors. Show that the number of ways to paint the sectors of the roulette into \(\leq p\) colours equals \(\sum_{d \mid n} \varphi(d) (x_d)^{n/d}\). (Two ways which can be transformed into each other by a rotation are considered equal. [Jablonski, 1892].)

**18. Necklaces with two colours.** Let \(N\) be the set of \(n\) vertices of a regular polygon, \(n = |N|\). Let be given \(a\) blue beads and \((n - a)\) red beads, \(0 \leq a \leq n\). On each vertex a bead is placed, thus obtaining a necklace. Let \(P_a^n\) be the number of different necklaces. Two necklaces that can be transformed into each other by rotation, or reflection with respect to a diameter, or both, are not distinguished from each other. Then we have \(P_0^n = 1, P_1^n = n/2, P_2^n = n^2/12\) if \(n \equiv 0 \pmod{6}\) or \((n^2 - 2)/12\) if \(n \equiv 1 \pmod{6}\) or \((n^2 - 4)/12\) if \(n \equiv 2 \pmod{6}\) or \((n^2 + 4)/12\) if \(n \equiv 3 \pmod{6}\). Compute \(P_a^n\) and generalize. ([Durrande, 1816], [Gilbert, Riordan, 1961], [LaGrange, R., 1962b], [Moreau, 1872], [Riordan, 1958], p. 162, [Titsworth, 1964]).

**19. The number of unlabeled graphs.** Two graphs \(G\) and \(G'\) over \(N\) are called equivalent, or isomorphic if there exists a permutation \(\sigma\) of \(N\), which induces a map from the set of edges of \(G\) onto the set of edges of \(G'\). In other words, \(\exists \sigma \in \Sigma(N), \{x, y\} \in G \iff \{\sigma(x), \sigma(y)\} \in G'\). Each equivalence class, thus obtained, is called an unlabeled graph, abbreviated UG (graphs as we have seen on p. 61 are called labeled graphs, to distinguish them from the UG; their vertices are distinguishable). For instance, there are three UG's with 4 nodes and 3 edges: \(G_1, G_2, G_3\) (\(G_4\) is equivalent to \(G_1\)) (see Figure 47).
From now on, $N = [n] = \{1, 2, \ldots, n\}$. With each $\sigma \in S(n)$ we associate the permutation $\delta$ of $\Pi_2[n]$ defined for each pair $\{x, y\}$ by $\delta(x, y) = (\sigma(x), \sigma(y))$. The set of the $\delta$ forms together the ‘group of pairs’, denoted by $S^2([n]) \leq S([n])$, which has a cycle indicator polynomial $Z(S^2([n])) = Z_n(x_1, x_2, \ldots)$; denoted by $Z_n(x_1, x_2, \ldots)$.

(1) Show that the number $g_{n,k}$ of $\Gamma G$ satisfies $x_k g_{n,k} x_k = Z_n(1 + x, 1 + x^2, 1 + x^3, \ldots)$.

(2) For $\sigma \in S([n])$ of type $(c_1, c_2, \ldots)$, let $I_k[c_1, c_2, \ldots]$ be the number of $k$-orbits (in $\Pi_2[n]$) of $\delta$. Then, $Z_n(x_1, x_2, \ldots)$ equals:

\[
\frac{1}{n!} \sum_{c_1 + 2c_2 + \cdots = n} \frac{n!}{c_1! c_2! \cdots} x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n}
\]

(3) Show that $I_k[c_1, c_2, \ldots, c_n] = c_k + c_k/2 \left( k - 1 + (k/2) \right) + (1/2k) x \sum_{i,j} i j c_i(c_i - \delta_{ij})$, where $[i, j]$ is the LCM of $i$ and $j$, $\delta_{ij}$ the Kronecker symbol, and $(x^i)^n = x_i$, if $x$ is an integer, and $= 0$ otherwise, the summation being taken over all $(i, j)$ such that $1 \leq i \leq j \leq n$ and $[i, j] = k$. (This theorem, in this form, is due to [Oberschelp, 1967].)

The counting unlabeled graphs and digraphs is done in the fundamental paper by [Pólya, 1937], and also in [Harary and Read, among others]. Thus, $Z_2 = x_1$, $Z_3 = -1/(3!) (x_1^3 + 3x_1^2x_2 + 2x_3)$, $Z_4 = (1/4!) (x_1^6 + 9x_1^4x_2^2 + 8x_1^2x_2^4 + 6x_2^6 + 6x_2x_4)$, \ldots The first values of $g_{n,k}$ are:

<table>
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<tr>
<th>$n \setminus k$</th>
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<td>24</td>
<td>56</td>
<td>115</td>
<td>221</td>
<td>402</td>
<td>663</td>
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</tbody>
</table>

*20. The number of unlabeled $m$-graphs. Let us call any system of $m$-blocks (p. 7) of $N$ an $m$-graph of $N$. In particular, an ordinary graph is a 2-graph. Let $g_{n,m}$ be the total number of unlabeled $m$-graphs (in the sense of the previous exercise). Then, for fixed $m$, when $n \to \infty$:

\[
g_{n,m} \approx \frac{2^{n-1}}{n!} \left( 1 + \frac{2}{n-2} \right) (1 + o(1)) \frac{1}{n-1}
\]

*21. Rearrangements. This is a generalization of as well a permutation and a minimal path (p. 20). Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set with $n$ elements. A rearrangement of $X$, (abbreviated RA) is a word of $X$ (p. 18).

Thus, for instance, for $X = \{a, b, c\}$ the RA $f_1 = b \ a \ a \ b \ c \ \ c \ b \ c$ and $f_2 = c \ \ a \ a \ a \ c \ a \ c \ a$ are of specification (4, 4) and (4, 0, 3), respectively. For $c_1 = c_2 = \cdots = c_n = 1$, we get back the permutations of $X$. A RA can be represented as a minimal path in the euclidean $R^n$, which describes a process of counting ballots for an election with $n$ candidates. The word $f_1$ is shown in Figure 48. (1) The number of $(c_1, c_2, \ldots, c_n)$-RA equals $(c_1, c_2, \ldots, c_n)$ (p. 27). (2) A sequence of $f \in X(c_1, c_2, \ldots, c_n)$ is a maximal row of consecutive $x_i$ in $f$, in $[n]$. For instance, $f_1$ has 7 sequences. What is the number of the $f \in X(c_1, c_2, \ldots)$ having $s$ sequences ([*David, Barton, 1962], p. 119)? (3) Compute $f_{l_1, l_2, \ldots, l_k}(c_1, c_2, \ldots, c_n)$, which is the number of the $(c_1, c_2, \ldots, c_n)$-RA such that between two letters $x_i$ there are at least $l_i$ other letters. (A generalization of [8d], p. 21, and Exercise 1, p. 198.) (4) If $X = [n]$, then we can consider $f$ as a map from $[p]$, $p := c_1 + c_2 +$
+ … + c_n into [n] such that for all i ∈ [n], |f^{−1}(i)| = c_i. (Figure 49 shows f_1 = 2 1 2 3 2 3 3 2.) An inversion of f is a pair (i, j) such that 1 ≤ i < j ≤ n and f(i) > f(j). (f_1 has 7 inversions.) Show that the number b(c_1, c_2, ..., c_n; k) of (c_1, c_2, ..., c_n)-RA of [n] with k inversions, c_i, c_2, ..., c_n ≥ 1, has for GF ∑b(c_1, c_2, ..., c_n; k)u^k the following rational fraction:

\[
\frac{(1 - u)(1 - u^2) \cdots (1 - u^n)}{\prod_{i=1}^{c_1} (1 - u^i) \cdot \prod_{i=1}^{c_2} (1 - u^{2i}) \cdots \prod_{i=1}^{c_n} (1 - u^{ni})}.
\]

(For c_1 = c_2 = ... = 1, we recover [4h], p. 230.) (5) We call the sum T(f) of the indices j ∈ [p − 1] such that f(j) > f(j + 1) (f is a (c_1, c_2, ..., c_n)-RA of [n]) the index of f. So the index is the sum of the j where there is a descent (or fall). Show that the number of RA for which T(f) = k equals b(c_1, c_2, ..., c_n; k). ([MacMahon, 1913, 1916] gives a proof using the GF; [Foata, 1968] and [*Cartier, Foata, 1969] give a ‘bijective’ proof.) (6) An ascent (or rise) of a (c_1, c_2, ..., c_n)-RA of [n], f, is an index i such that f(i) < f(i + 1). Compute the number A(c_1, c_2, ..., c_n; k) of the RA with (k − 1) ascents. (These numbers are a generalization of the Eulerian numbers [5c], p. 242. They give the solution to the problem of Simon Newcomb (p. 246).)

*22. Folding a strip of stamps. Given a strip of n stamps labelled 1, 2, ..., n from left to right, the problem is to determine the number A(n) of ways this strip can be folded along the perforations to that the stamps are piled one on top of each other without destroying the continuity of the strip. It is supposed that stamp labelled 1 has its front side facing the top of the pile and its left edge on the left as we look down on the pile. So A(1) = 1, A(2) = 2, and A(3) = 6 as it is shown by the following figures:

\[
\begin{align*}
\begin{array}{c}
1 \\
1
\end{array} \\
\begin{array}{c}
1 \\
1
\end{array} \\
\begin{array}{c}
1 \\
1
\end{array} \\
\begin{array}{c}
1 \\
1
\end{array} \\
\begin{array}{c}
1 \\
1
\end{array} \\
\begin{array}{c}
1 \\
1
\end{array}
\end{align*}
\]

\[c_4\] use the number d_3(m, k) of permutations of [m] with k orbits all ≥ 3 (See Exercise 7 p. 256). The first values of c_q are (for q ≤ 20, see [Wrench, 1968]):

\[
\begin{array}{c|cccccccc}
q & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
c_q & 1 & 1 & 139 & 571 & 161379 & 5246819 & 534703531 \\
\end{array}
\]

If n ≥ 2, prove that A(n) = 2an(n), where a(n) is a positive integer. Here are the known values of a(n):

\[
\begin{array}{cccccccc}
\hline
n & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
\hline
a(n) & 1 & 1 & 2 & 5 & 13 & 44 & 171 & 776 & 3636 & 17419 & 87707 & 456127 & 2429301 & 13848576 & 75523745 \\
\hline
\end{array}
\]

(Up to 10: [Touchard, 1950, 1952]; up to 12: [Sade, 1949a]; up to 16: [Koehler, 1968]; up to 28: [Lunnon, 1973].)

*23. An explicit and combinatorial Stirling expansion for the gamma function of large argument. Using the Watson lemma for Laplace transforms, show that

\[
\Gamma(x) \approx \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}} \left(1 + \sum_{q \geq 3} \frac{c_q}{x^q}\right),
\]

\[x \to \infty,\]

where the coefficients

\[
c_q = \sum_{k=1}^{q} (-1)^k \frac{d_3(2q + 2k, k)}{(2q + 3)(q + k)!}
\]

use the number d_3(m, k) of permutations of [m] with k orbits all ≥ 3 (See Exercise 7 p. 256). The first values of c_q are (for q ≤ 20, see [Wrench, 1968]):

\[
\begin{array}{c|cccccccc}
q & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
c_q & 1 & 1 & 139 & 571 & 161379 & 5246819 & 534703531 \\
\end{array}
\]

\[\text{Signed P.}\]
EXAMPLES OF INEQUALITIES AND ESTIMATES

In the preceding chapters we have established explicit formulas for counting sets. The sets we wanted to count were of the following type: a finite set \( N \) with \( n \) elements is given, and then we studied sets of combinatorial objects bound to \( N \) that satisfied some additional conditions. If these conditions are not simple, then the explicit formula is usually not simple either, difficult to obtain, and little efficient. It can often be replaced advantageously by upper and lower bounds. Evidently, the closer these bounds fit, the better.

In most of the cases we want to determine conditions in the form of inequalities between certain parameters (integers) that guarantee the existence or non-existence of configurations between these parameters. The search for such inequalities has the charm of challenging problems, since there is no general rule for obtaining this kind of results.

In this chapter we give also an example of the use of probabilistic language, and, moreover, an asymptotic expansion of the most easy kind.

7.1. CONVEXITY AND UNIMODALITY OF COMBINATORIAL SEQUENCES

Just as in the case of functions of a real variable, it is interesting to know the global behaviour of combinatorial sequences of integers \( v_k \): monotony, convexity, extrema; this is a fertile source of inequalities, which are particularly useful in estimates.

In this respect we recall some definitions.

I. A real sequence \( v_k \), \( k=0,1,2,... \), is called convex on an interval \([a, b]\) (containing at least 3 consecutive integers) when:

\[ v_k \leq \frac{1}{2} (v_{k-1} + v_{k+1}), \quad k \in [a+1, b-1]. \]

It is called concave on \([a, b]\) if, in \([1a]\), \( \leq \) is replaced by \( \geq \). In the case where the inequalities are strict for all \( k \), \( v_k \) is called strictly convex or strictly concave. \([1a]\) is equivalent to \( \Delta^2 v_k := v_{k+2} - 2v_{k+1} + v_k \geq 0 \) for all \( k \in [a, b-2] \) (p. 13). The polygonal representation of \( v_k \) has hence the form of Figures 50a or b. For instance, \( v_k := \binom{n}{m} \), \( m \) fixed \( \geq 2 \), is strictly convex on \([m, \infty]\), because \( \Delta^2 v_k = \binom{n+2}{m} - 2\binom{n+1}{m} + \binom{n}{m} = \binom{n}{m-2} > 0 \).

II. A real sequence \( v_k, \ k=0,1,2,... \), is called unimodal if there exist two integers \( a \) and \( b \) such that:

\[ k \leq a \quad \Rightarrow \quad v_k \leq v_{k+1} \quad ; \quad v_{a-1} < v_a = v_{a+1} = \cdots = v_b > v_{b+1} \quad ; \]

\[ k \geq b+1 \quad \Rightarrow \quad v_k \geq v_{k+1}. \]

Figure 51a represents the polygon of a unimodal sequence in the case of a plateau (\( \Rightarrow a < b \)) with 4 points, and Figure 51b shows the case of a peak (\( \Rightarrow a = b \)).

III. A real sequence \( v_k \geq 0, \ k=0,1,2,... \), is called logarithmically convex in \([a, b]\) if:

\[ \Delta^2 v_k := v_{k+2} - 2v_{k+1} + v_k \geq 0 \quad \text{for all} \quad k \in [a+1, b-1]. \]
It is called logarithmically concave if, in [1c], ≤ is everywhere replaced by ≥. In the case that the inequalities are strict for all k, v_k is called strictly logarithmically convex (or concave).

The terminology adopted here originates from the fact that [lc] is equivalent to saying that w_k := \log v_k is convex.

**THEOREM A.** Each sequence v_k(≥0) which is logarithmically concave on its interval of definition, say [a, b], is either nondecreasing or non-increasing or unimodal. Moreover, in the last case, if v_k is strictly logarithmically concave, then v_k has either a peak or a plateau with 2 points.

- v_k^2 \geq v_k-1v_k+1 can be written as v_k/v_k-1 ≥ v_k+1/v_k, which proves that z_k := v_k/v_k-1 is decreasing on [a+1, b], where a and b are supposed to be integers without loss of generality. If z_k ≥ 1 (or z_{k+1} ≤ 1), v_k is increasing (or decreasing) on [a, b]. If z_{k+1} > 1 and z_k < 1, v_k is evidently unimodal. In the last case, if v_k decreases strictly, then there is at most one value of k such that z_k = 1, which gives then a plateau of 2 points.

**THEOREM B.** If the generating polynomial:

\[ P(x) := \sum_{0 \leq k \leq p} v_k x^k, \quad v_p \neq 0, \]

of a finite sequence v_k(≥0), 0 ≤ k ≤ p, has only real roots (≤0), then:

\[ v_k^2 \geq v_{k-1}v_{k+1} \]

\[ \frac{k}{k-1} \frac{p-k+1}{p-k}, \quad k \in [2, p-1] \]

(this is one form of the Newton inequalities, [Hardy, Pólya, Littlewood, 1952], p. 104); hence v_k is unimodal, either with a peak or with a plateau of 2 points.

**THEOREM C.** The sequence of the absolute values of the Stirling numbers of the first kind, s(n, k), n fixed (≥3), k variable (≤n) is unimodal, with a peak or plateau of 2 points.

In fact, only the peak exists, [Erdős, 1953]; for estimates of its abscissa, see [Hammersley, 1951], [Moser, Wyman, 1958b].

**THEOREM D.** The sequence S(n, k) of the Stirling numbers of the second kind, n fixed (≥3), k variable (≤ n), is unimodal with a peak or plateau of 2 points. ([Harper, 1967], [Lieb, 1968]. See also [Bach, 1968], [Dobson, 1968], [Dobson, Rennie, 1969], [Harborth, 1968], [Kanold, 1968a, b], [Wegner, 1970], and Exercise 23, p. 296.)

**7.2. SPERNER SYSTEMS**

**DEFINITION.** A system \( \mathcal{P} \) of distinct blocks of a finite set \( N \), \( \mathcal{P} \subset \mathcal{B}(N) \).
is called a Sperner system, if for any two blocks, one is not contained in the other. In other words, if \( s(N) \) is the family of these systems:

\[
(\mathcal{S} \in s(N)) \iff ((B, B' \in \mathcal{S}) \Rightarrow (B \not\subset B' \text{ and } B' \not\subset B)).
\]

**THEOREM** [Sperner, 1928]. The maximum number of blocks of a Sperner system equals \( \binom{n}{[n/2]} \), where \([x]\) is the largest integer \( \leq x \).

For all \( \mathcal{S} \in s(N) \), we will prove with [Lubell, 1966]:

\[
\left( \sum_{B \in \mathcal{S}} \frac{1}{|B|} \right) \leq 1.
\]

This will imply the theorem, because \( \binom{n}{k} \leq \binom{n}{[n/2]} \) for all \( k \in [0, n] \), hence:

\[
\sum_{\mathcal{S} \in s(N)} \frac{1}{|\mathcal{S}|} \geq \sum_{\mathcal{S} \in s(N)} \frac{1}{\binom{n}{[n/2]}} = \frac{|s(N)|}{\binom{n}{[n/2]}}.
\]

From this we get, using [2a], \( |\mathcal{S}| \leq \binom{n}{[n/2]} \). This maximum value is reached by the Sperner system \( \mathcal{S}_{[n/2]}(N) \). We now prove [2a]. We introduce the name chain for a system \( \mathcal{C} = \{C_1, C_2, \ldots, C_n\} \) of \( N \), \( \mathcal{C} \subseteq \mathcal{P}(N) \) such that \( C_1 \subset C_2 \subset \cdots \subset C_n \), with strict inclusions. A chain is called maximal if it has a maximal number of blocks, namely \( n \). Let \( \mathcal{c}(N) \) be the family of maximal chains of \( N \). A maximal chain is evidently completely determined by the permutation \( (x_1, x_2, \ldots, x_n) \) of \( N \), given by: \( x_1 := C_1 \), \( x_2 := C_2 - C_1 \), \( x_n := C_n - C_{n-1} \). Hence \( |\mathcal{c}(N)| = n! \). Now we observe that a given system \( \mathcal{S} \) is a Sperner system if and only if each chain \( \mathcal{C} \in \mathcal{c}(N) \) satisfies \( |\mathcal{C} \cap \mathcal{S}| \leq 1 \). Let \( c_\mathcal{S} \) be the family of chains \( \mathcal{C} \in \mathcal{c}(N) \) such that \( |\mathcal{C} \cap \mathcal{S}| = 1 \). We define the map \( \varphi \) from \( c_\mathcal{S} \) into \( \mathcal{S} \) by \( \varphi(\mathcal{C}) := \) the unique block \( B \in \mathcal{C} \cap \mathcal{S} \). Of course \( \varphi \) is surjective, and for all \( B \in \mathcal{S} \), \( |\varphi^{-1}(B)| = |B|! (n - |B|)! \). It follows that:

\[
|c_\mathcal{S}| = \sum_{B \in \mathcal{S}} |\varphi^{-1}(B)| = \sum_{B \in \mathcal{S}} |B|! (n - |B|)!.
\]

It now suffices to combine \( |c_\mathcal{S}| \leq |\mathcal{c}(N)| = n! \) with [2b] to obtain [2a].

The number \( s(n) = |s(N)| \) of Sperner systems (unordered systems without repetition, in the sense of p. 3) is just, up to 2 units, the number of elements of a free distributive lattice with \( n \) generators, or, the number of monotone increasing Boolean functions with \( n \) variables. Since [Dedekind, 1897] numerous efforts have been made to compute or estimate this number [Agnew, 1961], [Gilbert, 1954], [Rivière, 1968], [Yamamoto, 1954]. Actually, the known values are:

\[
\begin{array}{ccccccccc}
\hline
n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
s(n) & 1 & 4 & 18 & 166 & 7579 & 782352 & 241468204996 \\
\hline
\end{array}
\]

\( s(5) \) due to [Church, 1940], \( s(6) \) due to [Ward, 1946], \( s(7) \) due to [Church, 1965]). The following upper and lower bounds hold:

\[
2^{\left( \binom{n}{[n/2]} \right)} \leq s(n) \leq 3^{\left( \binom{n}{[n/2]} \right)}
\]

([Hansen, 1967]) and also the asymptotic equivalent \( \log_2 s(n) \sim \binom{n}{[n/2]} \) ([Kleitman, 1969], [Shapiro, 1970]). Various extensions of the Sperner theorem have been suggested ([Chao-Ko, Erdős, Rado, 1961], [Hilton, Milner, 1967], [Katona, 1966, 1968], [Kleitman, 1968b], [Meshalkin, 1963], [Milner, 1968]).

### 7.3. Asymptotic Study of the Number of Regular Graphs of Order Two on \( N \)

**1. Graphical and geometrical formulation of the problem**

A regular graph of order \( r \) (integer \( \geq 0 \)) is a graph on \( N \), \(|N| = n\), such that there are \( r \) edges adjacent to every node \( x \in N \). Let \( G(n, r) \) be the number of these graphs. Evidently \( G(n, 0) = 1 \). For computing \( G(n, 1) \), observe that giving a regular graph of order 1 is equivalent to giving a partition of \( N \) into disjoint pairs (the edges). Hence \( G(2m+1, 1) = 0 \) and \( G(2m, 1) = (2m)!/(2^m m!) \). We investigate now \( G(n, 2) = g_n \). First, we give a geometric interpretation to these numbers ([Whitworth, 1901], p. 269, Exercise 160).

Let be given a set \( \Delta \) of \( n \) straight lines in the plane, \( \delta_1, \delta_2, \ldots, \delta_n \), lying...
in general position (no two among them are parallel, and no three among them are concurrent). Let \( P \) be the set of their points of intersection, \(|P| = \binom{n}{2}\). We call any set of \( n \) points from \( P \) such that any three different points are not collinear, a cloud. An example is shown in Figure 52, for

![Fig. 52.](image)

the case \( n=4 \), \( \{a, b, d, e\} \). Let \( \mathcal{G}(A) \) stand for the set of clouds of \( A \), then we have:

\[
\begin{align*}
3a] & \quad N \in \mathcal{G}(A) \iff N \subseteq P; \quad |N| = n; \quad \{(a, b, c) \in P, \delta \in A\} \Rightarrow \{a, b, c\} \cup \delta.
\end{align*}
\]

Giving a cloud is hence equivalent to giving a regular graph of order 2: it suffices to identify the lines \( \delta_1, \delta_2, \ldots, \delta_n \) with the nodes \( x_1, x_2, \ldots, x_n \) of \( N \), and each point of intersection \( \delta_i \cap \delta_j \) with the edge \( \{x_i, x_j\} \).

For example, with 3 points, we can get only 1 cloud; with 4 points, we have 3 clouds, since the clouds in \( \{\delta_1, \delta_2, \delta_3, \delta_4\} \) (Figure 52) are the sets \( \{a, b, d, e\}, \{a, c, d, f\} \) (\( b, c, e, f \)). The problem is to determine the number \( g_n = |\mathcal{G}(A)| \) of clouds of \( A \).

(II) A recurrence relation ([Robinson, 1951, 1952], [Carlitz, 1954b, 1960b]). Let now \( M := \{a_1, a_2, \ldots, a_{n-1}\} \) be a cloud of \( \Gamma := \{\delta_1, \delta_2, \ldots, \delta_{n-1}\} \). It is clear, by \([3a] \), that every straight line \( \delta_i, i \in [n-1] \), contains exactly two points of \( M \). Now we add an \( n \)-th line \( \delta_n \), so we obtain \( A := := \{\delta_1, \delta_2, \ldots, \delta_{n-1}, \delta_n\} \). We consider then an arbitrary point \( a_i \) of \( M \), which belongs to 2 lines, say \( \delta_a \) and \( \delta_i \) (or \( \Gamma \)), that intersect \( \delta_n \) in the points \( u \) and \( v \). (Figure 53) It is easily seen that \( N := \{a_1, a_2, \ldots, a_{i-1}, a_i, \ldots, a_{n-1}, u, v\} \) is a cloud of \( A \). Thus, if we let \( a_i \) run through the set \( a_1, a_2, \ldots, a_{n-1} \), we associate with every cloud \( M \in \mathcal{G}(\Gamma) \) a set \( \Phi(M) \) of \( (n-1) \) clouds of \( A \):

\[
\begin{align*}
3b] & \quad \Phi(M) \subseteq \mathcal{G}(A), \quad |\Phi(M)| = n - 1.
\end{align*}
\]

On the other hand, each cloud \( N \in \mathcal{G}(\Gamma) \) obtained in the preceding way (Figure 54) is obtained in one way only:

\[
\begin{align*}
3c] & \quad M, M' \in \mathcal{G}(\Gamma), \quad M \neq M' \Rightarrow \Phi(M) \cap \Phi(M') \neq \emptyset.
\end{align*}
\]

But in this way \( \mathcal{G}(\Gamma) \) is not completely obtained, because there exist singular clouds \( N \) of \( A \) that do not belong to any \( \Phi(M) \), for instance, the cloud shown in Figure 55. Let \( \mathcal{S} \) be the set of singular clouds of \( A \). Giving a cloud \( \in \mathcal{S} \) is evidently equivalent to giving a pair \( \{u, v\} \) among the \( (n-1) \) points of \( \delta_n \), and to giving a cloud on the \( (n-3) \) lines \( \delta_i \) that do not pass through \( \{u, v\} \). Hence:

\[
\begin{align*}
3d] & \quad |\mathcal{S}| = g_{n-3} \binom{n-1}{2}.
\end{align*}
\]
Now, according to [3c] we have the division:
\[ \mathcal{G}(d) = \left( \sum_{M \in \mathcal{M}(d)} \Phi(M) \right) \| \mathcal{G} ; \]
this gives, after passing to the cardinalities (using [3b] for (*)):
\[ g_n = |\mathcal{G}(d)| = \sum_{M \in \mathcal{M}(d)} |\Phi(M)| + |\mathcal{G}| =
\]
\[ (n - 1) |\mathcal{G}(F)| + |\mathcal{G}|. \]

Finally, by [3d]:
\[ [3e] \quad g_n = (n - 1) g_{n-1} + \left( \frac{n - 1}{2} \right) g_{n-3}, \]
\[ n \geq 3; \quad g_0 = 1, \quad g_1 = g_2 = 0. \]

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</table>

(III) A generating function

Using [3c] for (*), we get:

\[ [3f] \quad g(t) := \sum_{n \geq 0} \frac{g_n t^n}{n!} = 1 + \sum_{n \geq 3} \frac{g_n t^n}{n!} \]
\[ = 1 + \sum_{n \geq 3} (n - 1) g_{n-1} \frac{t^n}{n!} + \sum_{n \geq 3} \left( \frac{n - 1}{2} \right) g_{n-3} \frac{t^n}{n!} . \]

Taking the derivative of [3f] with respect to \( t \):
\[ g'(t) = \frac{t^{n-2}}{(n-2)!} + \frac{t^{n-3}}{2} \sum_{n \geq 3} \frac{g_{n-2}}{(n-2)!} + \frac{t^{n-3}}{2} \sum_{n \geq 3} \frac{g_{n-3}}{(n-3)!} . \]
\[ = t g'(t) + \frac{t^2}{2} g(t) . \]

Thus, considering \( g(t) \) as a function defined in a certain interval (to be specified later), we obtain the differential equation \( g'(t)/g(t) = t^2/(1-t) \), which gives, by integration on \((-1, +1)\) and exponentiation, and ob-

serving that \( g(0) = g_0 = 1 \).

\[ [3g] \quad g(t) := \sum_{n \geq 0} \frac{g_n t^n}{n!} = \frac{1}{\sqrt{1 - t}} \exp \left( - \frac{t^2 + 2t}{4} \right) . \]

(IV) The asymptotic expansion

We will use the 'method of Darboux' ([Darboux, 1878]) which is stated below. No proof will be given.

**Theorem.** Let \( g(z) = \sum_{n \geq 0} a_n z^n/n! \) be a function of the complex variable \( z \), regular for \( |z| < 1 \), and with a finite number \( l \) of singularities on the unit circle \( |z| = 1 \), say \( e^{i\xi_1}, e^{i\xi_2}, \ldots, e^{i\xi_l} \). We suppose that in a neighbourhood of each of these singularities \( e^{i\xi_n} \), \( g(z) \) has an expansion of the following form:

\[ [3h] \quad g(z) = \sum_{p \geq 0} c_p^{(k)} (1 - ze^{-i\xi_p})^{\alpha_p + pb_p}, \quad k \in [l] , \]

where the \( \alpha_p \) are complex numbers, and all \( b_p > 0 \). The branch chosen for each bracketed expression is that which is equal to 1 for \( z = 0 \). Under these circumstances, \( g_n \) has the following asymptotic expansion \( (n \to \infty) \):

\[ [3i] \quad g_n = \sum_{0 \leq p \leq \ell} \left\{ \sum_{1 \leq \xi \leq l} c_{\xi}^{(p)} (a_k + p b_k) \right\} (1 - e^{-i \xi})^{\alpha + pb} + O(n^{-\eta} n!) . \]

In [3i], \( \xi(q) \) is the smallest integer \( \geq \max_{1 \leq \xi \leq l} |b_{\xi} - 1| (q - \Re(a_k) - 1) \), and \( O(n^{-\eta} n!) \) means a sequence \( v_n \) such that \( v_n (n^{-\eta} n!) \) is bounded for \( n \to \infty \).

It is important to observe that formally the asymptotic expansion [3i] of \( g_n \), up to the \( O \) term, can be obtained by gathering for each singularity \( e^{i\xi_p} \) the coefficient of \( z^n/n! \) in [3h].

We apply this theorem to the function \( g(z) \), defined by [3g]; the only singularity is in \( z = 1 \). The expansion [3h] can be obtained using the Hermite polynomials \( H_n(x) \), [14n] (p. 50). Thus, if we put \( u := 1 - z \):

\[ g(z) = e^{-3/4} u^{-1/2} \exp \left( u - \frac{u^2}{4} \right) = e^{-3/4} u^{-1/2} \sum_{p \geq 0} H_p(1) 2^p! u^p \]
\[ = e^{-3/4} (u^{-1/2} + u^{1/2} + \frac{u^{3/2} - uv^{1/2} + \cdots}) . \]

Hence, by [3i], where \( l = 1, \quad e^{i\xi_p} = 1, \quad c_p^{(1)} = c_p = H_p(1) 2^p! \), \( a = -\frac{1}{2}, \quad b = 1, \quad \eta = 1 \).
\[ \xi(q) = q - 1 \text{ for all integers } q \geq 1, \text{ we get the asymptotic expansion of } g_n: \]

\[ g_n = e^{-3/4} \sum_{p=0}^{q-1} \left( \frac{H_p(1)}{2^np} \right) n^{-q}, n \to \infty. \]

Taking into account the Stirling formula \( n! = n^{n-\frac{1}{2}} 2\pi n (1 + O(n^{-1})) \), [3j] gives us, if we take only the first term \( (q=1) \):

\[ g_n = e^{-3/4} \sqrt{2n} n^{n-\frac{1}{2}} \left( 1 + O(n^{-1/2}) \right) \sim e^{-3/4} \sqrt{2n} n^{n-\frac{1}{2}}. \]

(V) Direct computation

We could have determined \( g_n \) directly, by an argument analogous to that on p. 235. It is the number of symmetric and antireflexive relations on \([n]\) such that each section has 2 elements. Hence:

\[ g_n = G(n, 2) = \sum_{\omega_1 \leq \omega_2 \leq \ldots \leq \omega_n} (1 + w_i w_j) \]

from which follows, after some computations:

\[ g_n = G(n, 2) = \frac{1}{2^{n-1}} \sum_{\omega_1 < \omega_2 < \ldots < \omega_n} (-1)^{\omega_2 + \beta_1} \times \]

\[ \times (2\omega_1)! (2\omega_2 + \beta_1)! \frac{1}{2^{\omega_1}} \binom{n}{\omega_1}. \]

(which leads to the GF [3g] and conversely).

(VI) The general case

The explicit computation of \( G(n, r) \) (p. 273) can also be done by:

\[ G(n, r) = \sum_{\omega_1 \leq \omega_2 \leq \ldots \leq \omega_r} (1 + w_i w_j), \]

but the formulas become very quickly extremely complicated. Thus, \( G(2m+1, 3) = 0 \) and

\[ G(2m, 3) = \sum_{\omega_1 < \omega_2 < \omega_3} (-1)^{\omega_2 + \beta_1} \times \]

\[ \times (2m)! (2\omega_1)! \frac{(2m)!}{\omega_1! \omega_2! \beta_1! (\omega_1 + \omega_2 - m)!}, \]

where \( \omega_1 + 2\omega_2 + 3\omega_3 + \beta_1 = 3m \) and \( \omega_1 + \omega_2 > m \). The first values of \( G(n, r) \) are:
from which we obtain the GF of the cumulants of $C_n$:

$$[4e] \quad \gamma(t) = \log \{g(e^t)\} = \sum_{1 \leq i \leq n} \log \left(1 + \frac{e^t - 1}{i}\right).$$

We expand $[4e]$ using $[2a]$ (p. 206) for $(e^t)$; then we obtain:

$$[4f] \quad \sum_{n \geq 0} \kappa_n \frac{e^n}{n!} = \gamma(t) = \sum_{1 \leq i \leq n} \left\{ \sum_{j=1}^{i-1} \frac{(-1)^{j-1}}{i} \left(\frac{e^t - 1}{i}\right)^j \right\} \equiv \sum_{1 \leq i \leq n} \left\{ \sum_{j=1}^{i-1} \frac{(-1)^{j-1}}{i^j} (i-1)! \sum_{\pi \in \iota_i} S(m, l) \frac{i^m}{m!} \right\},$$

and by identifying the coefficients of $t^m/m!$ in $[4f]$.

**Theorem A.** The cumulants of the RV $C_n$ defined by $[4b]$ equal:

$$[4g] \quad \kappa_n = \kappa_n(C_n) = \sum_{1 \leq i \leq \ell \leq n} \left\{ (-1)^{\ell-1} (i-1)! S(m, l) \zeta_\ell(i) \right\},$$

where $S(m, l)$ is the Stirling number of the second kind and

$$[4h] \quad \zeta_\ell(i) := \sum_{1 \leq i \leq \ell \leq n} \frac{1}{i} = 1 + \frac{1}{2} + \cdots + \frac{1}{\ell}.$$  

Thus, by passing to the moments:

$$\mu_1 = \mu_1(C_n) = \kappa_1 = \zeta_1(1);$$

$$[4i] \quad \mu_2 = \text{var } C_n = D^2(C_n) = \kappa_2 = \zeta_2(1) - \zeta_2(2).$$

For studying the behaviour of the limit of $C_n$, we state the central limit theorem (in very general form due to [Lindeberg, 1922]; see, for instance, [*Renyi, 1966], p. 412-21, for a proof):

**Theorem B.** Let $X_{n,i}$ be a double sequence of RV, defined for $n \in \mathbb{N}$ and $1 \leq i \leq k_n$, where $k_n$ are given integers $> 0$. We suppose that the variables $X_{n,i}$, $n$ fixed, $i$ variable, $i \in [k_n]$, are independent, which is formulated by saying 'the $X_{n,i}$ are row-independent'. If we define new RV $S_n$ and $Y_{n,i}$ by:

$$[4j] \quad S_n := \sum_{1 \leq i \leq k_n} X_{n,i}, \quad Y_{n,i} := \frac{X_{n,i} - E(X_{n,i})}{D(S_n)},$$

with, for distributions function of $Y_{n,i}$:

$$[4k] \quad G_{n,i}(y) := P(Y_{n,i} < y),$$

then the condition $[4j]$ (of Lindeberg):

$$[4l] \quad \forall \varepsilon > 0, \lim_{n \to \infty} \sum_{1 \leq i \leq k_n} \int_{|y| > \varepsilon} y^2 dG_{n,i}(y) = 0$$

implies $[4m]$ (central limit theorem):

$$[4m] \quad \lim_{n \to \infty} P\left( \frac{S_n - E(S_n)}{D(S_n)} < x \right) = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt.$$

The conclusion $[4m]$ still holds when $E(S_n)$ and $D(S_n)$ are replaced by equivalent ones, when $n \to \infty$.

The role of the RV $S_n$ will be played by $C_n$ $[4j]$, for our application. Thus, we have to interpret $C_n$ as a sum $[4j]$. To do this, we define the sequence $X_{n,i}$ of row-independent RV, $1 \leq i \leq n$, by:

$$[4n] \quad P(X_{n,i} = 1) = 1/i, \quad P(X_{n,i} = 0) = 1 - 1/i.$$

The GF of the probabilities of the $X_{n,i}$ equal $g_{X_{n,i}}(u) = (i - 1 + u)/i$. Thus we get, by $[4d]$ for $(*)$, and by the row-independence for $(**)$:

$$\vartheta_{X_{n,i}}(u) = \prod_{1 \leq i \leq n} g_{X_{n,i}}(u) = g_{X_{n,i}}(u),$$

from which follows:

$$[4o] \quad C_n = \sum_{1 \leq i \leq n} X_{n,i}.$$

Furthermore, we show that condition $[4i]$ is satisfied by the $X_{n,i}$. Because of $[4i]$:

$$D^2(C_n) = \sum_{2 \leq i \leq n} \frac{i-1}{i^2} > \sum_{2 \leq i \leq n} \frac{1}{i^2} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n} > \log n - \frac{1}{2} - \frac{1}{3} > \log n - 2.$$
Hence:

\[ |Y_{n,i}| = \frac{X_{n,i} - E(X_{n,i})}{D(C_n)} < \frac{1}{\sqrt{\log n - 2}} \leq \frac{1}{\sqrt{\log n - 2}}, \]

which, for \( n \) sufficiently large, implies \( |Y_{n,i}| < \varepsilon \), in other words, \( \int_{|y|>\varepsilon} y^3 dG_{n,i}(y) = 0 \), for all \( i \in [n] \); hence [41] follows. Finally, we use \( E(S_n) \sim \log n \) and \( D(S_n) \sim \sqrt{\log n} \) to obtain by [4m]:

\[
\lim_{n \to \infty} P \left\{ \frac{C_n - \log n}{\sqrt{\log n}} < x \right\} = \Phi(x).
\]

In other words ([*Feller, I, 1968], p. 258): "The number of permutations with a number of orbits between \( \log n + \alpha \sqrt{\log n} \) and \( \log n + \beta \sqrt{\log n} \), \( \alpha < \beta \), equals approximately \( n!\{\Phi(\beta) - \Phi(\alpha)\} \)."

We give, rapidly, another example of RV associated with random permutations. We will deal with \( J_n = I_n(\omega) \), the number of inversions of the permutation \( \omega \) (p. 237). The GF of the probabilities is ([4h], p. 239):

\[
y(t) = \sum_{n=1}^{\infty} \frac{1}{n!} \prod_{1 \leq i \leq n} \frac{1 - u^j}{1 - u} \cdot \frac{1 + u}{2} \cdot \frac{1 + u + u^2}{3} \cdot \frac{1 + u + u^2 + \cdots + u^{n-1}}{n};
\]

hence we get for the GF of the cumulants:

\[
\gamma(i) = \sum_{m=0}^{\infty} \frac{x_m}{m!} \gamma_m = \sum_{1 \leq j \leq n} \log \frac{e^j - 1}{j(e^j - 1)} - \sum_{1 \leq j \leq n} \log \left( \frac{1 + j}{2} + \frac{j^2}{3} + \frac{j^3}{4} + \cdots \right).
\]

By [5a] (p. 140) follows: \( x_m = \sum_{j=1}^{\infty} L_m(j/2, j^2/3, \ldots) \). Hence \( \mu_1 = E(I_n) = n(n-1)/4 \) (cf. p. 160), \( \mu_2 = D^2(I_n) = n(n-1)(2n+5)/12 \); in other words \( E(S_n) \sim n^2/4 \), \( D(S_n) \sim n^3/6 \).

The factorization [4p] suggests that we define the row-independent RV \( X_{n,i} \) by \( P(X_{n,i} = k) = 1/i \), where \( (k+1) \in [i] \), and then we prove easily that the Lindeberg condition [41] is satisfied. Thus:

\[
\lim_{n \to \infty} P \left\{ \frac{I_n - n^2/4}{n^{3/2}/2} < x \right\} = \Phi(x).
\]

In other words: "The number of permutations whose number of inversions lies between \( n^2/4 + a_n^3/6 \) and \( n^2/4 + b_n^3/6 \), \( a < b \), equals approximately \( n!\{\Phi(b) - \Phi(a)\} \}" ([*Feller, I, 1968], p. 257). For many other problems of random permutations, see [Gontcharoff, 1944] and [Shepp, Lloyd, 1966].

7.5. THEOREM OF RAMSEY

The Ramsey theorem generalizes the 'Dirichlet pigeon-hole principle': If \( n+1 \) objects are distributed over \( n \) pigeon holes, at least one pigeon hole contains at least two objects. It introduces a sequence of numbers whose computation and estimation are still among the most fascinating problems of combinatorial analysis.

(i) Statement of the 'bicolour theorem' and definition of the Ramsey numbers \( \rho(b; p, q) \)

DEFINITION. Let three integers be given, \( b, p, q, 1 \leq b \leq p, q \). A finite set \( N \) is called Ramsey \((b, p, q)\) if for all divisions \((\mathcal{E}, \mathcal{D})\) of \( \Psi_b(N) \) into two subsets, \( \mathcal{E} + \mathcal{D} = \Psi_b(N) \), (p. 25) \textit{at least one} of the following two statements is true:

\[ [5a] \quad \text{There exists a } P \text{ such that } P \in \Psi_p(N), \quad \Psi_b(P) \subseteq \mathcal{E}. \]

\[ [5b] \quad \text{There exists a } Q \text{ such that } Q \in \Psi_q(N), \quad \Psi_b(Q) \subseteq \mathcal{D}. \]

Now we can state the 'bicolour' theorem of Ramsey. It is called the 'bicolour' theorem, because a division into two subsets \( \mathcal{E} + \mathcal{D} \) is equivalent to colouring each block \( B \in \Psi_b(N) \) in one of two given colours, say, carmine and dove-gray.

THEOREM. There exists a triple sequence \( \rho(b; p, q) \) of integers \( b > 0 \), called bicolour \( b \)-ary Ramsey numbers (multicolour numbers will be investigated in Exercise 26, p. 298), which is characterized by the following property...
[5c] concerning an arbitrary finite set $N$:

$$N \text{ is Ramsey-}(b; p, q) \iff |N| \geq \rho(b; p, q).$$

Moreover:

$$\rho(b; p, q) \leq 1 + \rho(b - 1; \rho(b; p - 1, q), \rho(b; p, q - 1)).$$

([Ramsey, 1930]. Our exposition is an adaptation of [*Ryser, 1963], pp. 38–46.)

(II) Some special values of $\rho(b; p, q)$

First, it is clear that the roles of $p$, $\mathcal{C}$ and $q$, $\mathcal{D}$ are symmetric; so:

$$\rho(b; p, q) = \rho(b; p, q).$$

We also show:

$$\rho(1; p, q) = p + q - 1, \quad 1 \leq p, q.$$  

Let $N$ be a finite set, such that $|N| = n \geq p + q - 1$. Suppose a division of $\mathcal{P}_p(N) = \mathcal{N}$ into two subsets $\mathcal{C} + \mathcal{D} = N$ is given. Then we have $|\mathcal{C}| + |\mathcal{D}| - n \geq p + q - 1$, hence $|\mathcal{C}| \geq p$ or $|\mathcal{D}| \geq q$. If $|\mathcal{C}| \geq p$, there exists a $P \in \mathcal{P}_p(N)$ such that $\mathcal{P}_p(P) = \mathcal{C}$; if $|\mathcal{D}| \geq q$, there exists likewise a $Q \in \mathcal{P}_q(N)$ such that $\mathcal{P}_q(Q) = \mathcal{D}$. Thus, $N$ is Ramsey-(1; $p, q$) if $n \geq p + q - 1$.

Conversely, if $|N| < p + q - 1$, in other words, if $|N| - n \leq p + q - 2$, we only have to choose a division into two subsets $\mathcal{C} + \mathcal{D} = N$ such that $|\mathcal{C}| = n - p - 1$, $|\mathcal{D}| = q - 1$ to see that $N$ cannot be Ramsey-(1; $p, q$).

Finally, we prove:

$$\rho(b; h, q) = q \quad (= \rho(h; q, h)), \quad h \leq q.$$  

We first prove that each finite set $N$ such that $n = |N| \geq q$ is Ramsey-(b; $p, q$). For a division into two subsets $\mathcal{C} + \mathcal{D} = \mathcal{P}_b(N)$ there are two cases:

(I) $\mathcal{C} \neq \emptyset$. Then choose $P \in \mathcal{C}$; hence $|P| = p = b$ with implies hence evidently $\mathcal{P}_b(P) = \{P\} \subseteq \mathcal{C}$.

(II) $\mathcal{C} = \emptyset$. Then $\mathcal{D} = \mathcal{P}_b(N)$. Now, $n = |N| \geq q$. Hence $\mathcal{P}_b(N)$ is not empty, and we can choose $Q$ there. Necessarily $|Q| = q$ and $\mathcal{P}_b(Q) \subseteq \mathcal{P}_b(N) = \mathcal{D}$.

Conversely, if $|N| < q$, in other words, if $|N| - n \leq q - 1$, it suffices to choose the division into two subsets $\mathcal{C} + \mathcal{D} = \mathcal{P}_b(N)$ such that $\mathcal{C} = \emptyset$ to see (by $\mathcal{P}_b(N) = \emptyset$) that $N$ cannot be Ramsey-(b; $p, q$).

Taking into account [5e, f, g] we suppose from now on that:

$$\rho(b; p, q) = 1 < b < p, q.$$  

(III) Choice of the induction for $\rho(b; p, q)$

Let $\mathcal{R}(b)$ be the table of the values of the double sequence $\rho(b; p, q)$, $p, q \geq 1$, $b$ fixed $\geq 1$, extended by $\rho(b; p, q) = 0$ if not $1 \leq b \leq p, q$. We know already $\mathcal{R}(1)$, according to [5f]. To prove the existence of $\rho(b; p, q)$, $1 \leq c \leq b - 1$, we suppose the existence of all the tables $\mathcal{R}(c)$ where $b$ is fixed $\geq 2$ (existence of all the $\rho(c; p, q)$, with $c \leq b - 1, p, q$), as well as the existence of:

$$\rho(b; p, q) \leq 1 + \rho(b - 1; p', q'),$$  

in other words [5d], because of [5i].

(IV) Proof of the theorem of Ramsey

We observe that [5j] is equivalent to proving that every finite set $N$ that satisfies:

$$\rho(b; p, q) = 1 < b < p, q$$

is Ramsey-(b; $p, q$) ($p'$, $q'$ defined in [5i]).

Let $N$ be such that [5k] holds, and choose $x \in N$, and let $M := N \setminus \{x\}$; then, by [5k]:

$$|M| = n - 1 \geq 1 + \rho(b - 1; p', q').$$  

Now we associate with the division $\mathcal{C} + \mathcal{D} = \mathcal{P}_b(N)$ the division $\mathcal{C}' + \mathcal{D}' = \mathcal{P}_b(M)$, defined by:

$$\mathcal{C}' := \{C \setminus \{x\} \mid C \in \mathcal{C}\}, \quad \mathcal{D}' := \{D \setminus \{x\} \mid D \in \mathcal{D}\}.$$  

According to [5l], $M$ is Ramsey-($b - 1; p', q'$), which implies for $\mathcal{C}'$ and
There exists an $X$ such that $X \in \Psi_b(M)$, $\Psi_{b-1}(X) \subseteq \mathcal{D}'$.

We suppose now that we are in the case [5n]. Because $|X|=p'=\rho(b; p-1, q)$, the set $X$ is Ramsey-($b; p-1, q$); hence, we have for the division $\mathcal{G}'' + \mathcal{D}' = \Psi_b(X)$, defined by

$$\mathcal{G}'' := \mathcal{G} \cap \Psi_b(X), \quad \mathcal{D}' := \mathcal{D} \cap \Psi_b(X),$$

at least one of the following two possibilities:

There exists a $P'$ such that $P' \in \Psi_{p-1}(X)$, $\Psi_b(P') \subseteq \mathcal{G}''$.

There exists a $Q$ such that $Q \in \Psi_q(X)$, $\Psi_b(Q) \subseteq \mathcal{D}'$.

In the case [5r], evidently $Q \in \Psi_q(N)$, because $X \subseteq N$; hence $\Psi_b(Q) \subseteq \mathcal{D}$, since $\mathcal{D}' \subseteq \mathcal{D}$, [5p]. So we have proved [5b].

In the case [5q], we will show that the set $P = P' \cup \{x\}$ satisfies [5a] indeed, in other words, that $\Psi_b(P) \subseteq \mathcal{G}$. We put:

$$\mathcal{X}_0 := \{B \mid B \in \Psi_b(P), x \notin B\}, \quad \mathcal{X}_1 := \{B \mid B \in \Psi_b(P), x \in B\}.$$

Hence:

$$\mathcal{G} := \mathcal{X}_0 + \mathcal{X}_1.$$

With every graph $\mathcal{G}$ on $N$, we associate the following two numbers:

1. The number $c(\mathcal{G})$, which is equal to the maximum number of elements of a complete subgraph of $\mathcal{G}$; (2) the number $i(\mathcal{G})$, equal to the maximum number of elements of independent sets $\mathcal{I}$ (i.e. complete subgraphs of $\mathcal{G}$).

Let now be given two integers $p, q > 0$. We say that $\mathcal{G}$ is a $(p, q)$-graph if $c(\mathcal{G}) < p$ and $i(\mathcal{G}) < q$. This means that $\mathcal{G}$ (or $\mathcal{G}$) does not contain a complete subgraph of $p$ elements (or $q$ elements). Hence, the negation of [5c] (p. 284) can be written:

$$[6a] \text{there exists a}: \mathcal{G} \in \Psi_2(N) \iff |N| + 1 \leq \rho(p, q),$$

and the problem becomes that of constructing $(p, q)$-graphs with the largest number of vertices, thus providing a constructive procedure for obtaining lower bounds for the Ramsey numbers $\rho(p, q)$.

We will illustrate this with the computation of $\rho(3, 3)$. Inequality [5d] (p. 284) combined with [5f] ($b=2$) gives:

$$[6b] \rho(p, q) \leq \rho(p, q-1) + \rho(p-1, q).$$
This gives, together with \( \rho(2,3) = \rho(3,2) = 3 \) and [5g]:

\[
[6c] \quad \rho(3, 3) \leq 6.
\]

On the other hand, the graph \( \mathcal{G} \) of Figure 56 over \( N, |N| = 5 \), whose edges are indicated by full lines, does not contain any triangle (=complete subgraph of 3 elements); the complementary graph neither does \( \mathcal{G} \) is indicated by dotted lines). Hence, by [6a]:

\[
[6d] \quad \rho(3, 3) \geq 6.
\]

Together, [6c, d] imply \( \rho(3, 3) = 6 \).

Below the first values of \( \rho(p, q) \) that are either known or for which bounds are known. The table should be completed by symmetry (cf. [5e], p. 284). For \( \rho(3, 8) \), for instance, 27–30 means \( 27 < \rho(3,8) < 30 \).

<table>
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<th>( p )</th>
<th>2</th>
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<td>23</td>
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<td>26-37</td>
<td>39-44</td>
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<tr>
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<td>18</td>
<td>25-28</td>
<td>34-44</td>
<td>?-66</td>
<td>?-94</td>
<td>?-129</td>
<td>?-170</td>
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<td>51-94</td>
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<td>?-521</td>
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</tr>
<tr>
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<td>102-169</td>
<td>?-322</td>
<td>?-544</td>
<td>?-887</td>
<td>?-1371</td>
<td></td>
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</tr>
<tr>
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### 7.7. SQUARES IN RELATIONS

Let \( M \) be a finite set and \( a \) an integer, \( 1 \leq a \leq m = |M| \). Determine the smallest integer \( f = f(m, a) \), such that each \( k \)-relation \( \mathcal{R} \) on \( M, \mathcal{R} \subseteq M^2 (= M \times M) \), \( |\mathcal{R}| = k \geq f \), contains at least one \( a^2 \)-square ([Zarankiewicz, 1951]). This is a product set of the form \( AA' = A \times A' \), where \( A, A' \subseteq M, |A||A'| = a \). In other words, when \( \mathcal{E} = \mathcal{E}(a) \) is the set of \( a^2 \)-squares of \( M^2 \):

\[
[7a] \quad k \geq f(m, a) \iff \forall \mathcal{R} \subseteq \mathcal{E}(M^2), \exists (A, A') \in \mathcal{E}(M)^2, AA' \in \mathcal{R},
\]

where \( \mathcal{E}(M)^2 := \mathcal{E}(M) \times \mathcal{E}(M) \). Evidently \( a^2 \leq f \leq m^2 \).

We transform [7a] by introducing for each \( a^2 \)-square \( AA' \in \mathcal{E} \) the set of \( r(AA') \) of the \( k \)-relations on \( M \) that contain \( AA' \). Hence [7a] is equivalent with:

\[
[7b] \quad k \geq f(m, a) \iff \forall \mathcal{R} \subseteq \mathcal{E}(M^2), \exists (A, A') \in \mathcal{E}(M)^2, AA' \in \mathcal{R},
\]

This will provide us with a lower bound for \( f \).

**Theorem A.** There exists a constant \( c_1 = c_1(a) > 0 \) independent of \( m \) such that:

\[
[7c] \quad f(m, a) > c_1 m^2 \cdot m^{-3/a}.
\]

In fact, [7b] implies, by [7d] (p. 194):

\[
[7d] \quad |\mathcal{E}(M^2)| = \sum_{(A, A') \in \mathcal{E}(M)^2} |r(A, A')|.
\]

Now:

\[
[7e] \quad |\mathcal{E}(M^2)| = \binom{m^2}{k}, \quad |\mathcal{E}(M)| = \binom{m^2}{a},
\]

\[
|r(A, A')| = \binom{m^2 - a^2}{k - a^2}.
\]

Hence [7b] becomes, by [7d, e]:

\[
[7f] \quad k \geq f(m, a) \iff \binom{m^2}{k} \leq \binom{m^2}{a} \binom{m^2 - a^2}{k - a^2}.
\]

We weaken (*) by using: (1) (*) \( \Rightarrow (*)_0 \); (2) \( m a \leq m^2 \), for \( (***): \) (3) \( m^2/k < (m^2 - 1)/(k - 1) \) for \( (****): \)

\[
[7g] \quad \binom{m^2}{k} \leq \frac{m^2}{k(k-1)} \ldots \frac{m^2 - a^2 + 1}{k(k-1) \ldots (k-a^2 + 1)} \leq \binom{m^2}{a} \binom{m^2 - a^2}{a - 1} \binom{m^2 - a^2}{a - 1}.
\]

Hence, by [7f, g]:

\[
[7h] \quad k \geq f(m, a) \iff \binom{m^2}{k} < \frac{m^2}{a} \Rightarrow k > (a)^{3/2}, m^2, m^{-3/2},
\]

which is [7c].

**Theorem B.** There exists a constant \( c_2 = c_2(a) > 0 \) independent of \( m \) such that:

\[
[7i] \quad f(m, a) < c_2 m^2 \cdot m^{-3/a}.
\]
Let $M = [m] = \{1, 2, \ldots, m\}$ and

$$r_j := \lfloor \mathfrak{R}_j \rfloor \quad (\text{hence } \sum_{j=1}^{m} r_j = k),$$

where $\lfloor \mathfrak{R}_j \rfloor$ means the second section of $\mathfrak{R}$ by $j$ (see Figure 57). Clearly $N$ contains a $a^2$-square, if there exists an $a$-block $A(\subset M)$ which is contained in at least $a$ of the subsets $\{\mathfrak{R}_j\}$ of $M, j \in [m]$. Now, according to an argument analogous to the 'pigeon-hole principle' (p. 91), this happens as soon as:

$$\sum_{j=1}^{m} \left( \frac{r_j}{a} \right) > (a - 1) \binom{m}{a} \quad (\Rightarrow k > f(m, a)).$$

We now must majorize $k$ as good as possible, using [71, j]. (For a more precise statement; see [Znám, 1963, 1965], [Guy, Znám, 1968].) By convexity of the function $\frac{x}{a}$ for $x > a$ (its second derivative is always positive: $d^2(x/a)/dx^2 = 2x \sum_{i=1}^{a-1} (x-i)(x-j)$ and the related Jensen inequality, we obtain, using [71] for $\star$:

$$\sum_{j=1}^{m} \left( \frac{r_j}{a} \right) > m \left( \frac{r_1 + r_2 + \cdots + r_m}{a} \right)^a = m \left( \frac{k}{a} \right)^a > m \left( \frac{(k/m) - a}{a} \right)^a \left( \frac{m}{a} \right)^a.$$ 

Consequently, by $\binom{m}{a} < m^a/a!$ for $\star\star$:

$$\sum_{j=1}^{m} \left( \frac{r_j}{a} \right) > (a - 1) \binom{m}{a} \quad (\Rightarrow k > am + (a - 1)^{1/a} \cdot m^{2 \cdot m^{-1/a}} \Rightarrow k > f(m, a)).$$

Hence $f(m, a) < am + (a - 1)^{1/a} \cdot m^{-1/a}$, which implies [71].

The following is a table of the known values of $f(m, a)$. (See all the quoted papers by Guy and Znám, and Exercise 29, p. 300.)

<table>
<thead>
<tr>
<th>$a\setminus m$</th>
<th>2 4 7 10 13 17 22 25 30 35 40 46 53 57 61</th>
<th>$\rightarrow$</th>
</tr>
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<tbody>
<tr>
<td>2</td>
<td>4 7 10 13 17 22 25 30 35 40 46 53 57 61</td>
<td>$\rightarrow$</td>
</tr>
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</table>

It has been proved that $f(m, 2) \sim m^{3/2}$, $m \to \infty$ ([Čulik, 1956], [Hyltén-Cavallius, 1958], [Kővari, Sós, Turán, 1954], [Reiman, 1958], [W. G. Brown, 1966]), but no asymptotic expression is known for $f(m, a), a \geq 3, \text{fixed}, m \to \infty$. A conjecture is that there exist constants $c(a) > 0$ such that $f(m, a) \sim c(a) m^2 m^{-1/a}$. 

**S U P P L E M E N T A N D E X E R C I S E S**

1. **Vertical convexity of Stirling numbers and Bell numbers.** (for a generalization of these properties see [Comtet, 1972].) (1) Show that for fixed $k$, the sequence $S(n, k)$ is convex, $n > k$. Same question for $s(n, k)$. (2) The sequence of numbers $w(n)$ of partitions of a set with $n$ elements (p. 210) is convex.

2. **Subsequences of the Pascal triangle.** The sequence $a_n := \binom{2n}{n}$ is convex. Does $A^k a_n > 0$ for $k \geq 3$ also hold? Analogous questions for $\binom{2n+c}{n}$ and $\binom{bn}{an}$, $a, b, c$ integer, $1 \leq a \leq b$. For $a$ and $b$ integers $\geq 1$, and $n \to \infty$, we
have:
\[
\binom{n}{a+b} \sim \frac{(a+b)^{(n+a+b)+1/2}}{a^{a+b+1/2}b^{b+a+1/2}} \cdot \sqrt{2\pi n}.
\]

[Use the Stirling formula \(n! \sim (n/e)^n \sqrt{2\pi n}\).]

3. **Unimodality of the Eulerian numbers.** Show that the Eulerian polynomials \(A_n(u)\) (p. 244) form a Sturm sequence, that is, \(A_n(u)\) has \(n\) real roots \((\leq 0)\), separated by the roots of \(A_{n-1}(u)\). [Hint: Use the recurrence relation \(A_n(u) = (u-u^2)A_{n-1}(u) + nuA_{n-2}(u)\).] Use this to prove that the sequence \(A(n, k)\), for fixed \(n\), is unimodal.

4. **Minimum of a partition of integers function.** With every partition \((Y) = (Y_1, Y_2, \ldots, Y_m)\) of \(n\) into \(m\) summands \(y_1 + y_2 + \cdots + y_m = n\), \(y_1 \geq y_2 \geq \cdots \geq y_m \geq 1\), we associate \(W(Y) = \sum_{i=1}^{m} \binom{y_i}{k}\). Then, for \(m, n, k\) fixed, the minimum of \(W(Y)\) occurs for a partition \((Y)\) that satisfies \(y_i - y_j < 1\) for all \((i, j)\) such that \(1 \leq i < j \leq m\).

5. **The most agglomerated system.** Let \(N\) be a set, and \(\mathcal{S}\) a system of \(N\) consisting of \(k\) (distinct) blocks all with \(b(\geq 1)\) elements, \(\mathcal{S} \in \mathcal{P}_b(B(N))\). Then \(M = \bigcup_{B \in \mathcal{S}} B\) has for minimal number of elements the smallest integer \(m\) such that \(k \leq \binom{m}{b}, (b, k\text{ fixed})\).

6. **Partition into unequal blocks.** The maximum number of blocks of a partition of \(N, |N| = n\), into blocks with all different numbers of elements equals the largest integer \(\leq (1/2)(-1 + \sqrt{8n+1})\).

7. **Bounds for \(S(n, k)\).** The inequalities \(k^{n-k} \leq S(n, k) \leq \binom{n-1}{k-1}k^{n-k}\) follow from [2c] (p. 207). Improve these bounds for the Stirling numbers of the second kind.

8. **The number of \(k\)-Sperner systems.** The number \(s(n, k)\) of Sperner systems with \(k\) blocks of \(N, |N| = n\), satisfies \(s(n, 2) = (1/2)!(4^n - 2.3^n + 2^n)\), \(s(n, 3) = (1/31)!(8^n - 6.6^n + 6.5^n + 3.4^n - 6.3^n + 2.2^n)\), \(s(n, 4) = (1/41)!(16^n - 12.12^n + 24.10^n + 4.9^n - 18.8^n + 6.7^n - 36.6^n + 36.5^n + 11.4^n - 22.3^n + 6.2^n)\) ([Hillman, 1955]). Determine for \(s(n, k)\) an explicit formula of minimal rank.

9. **Asymptotic expansion of the Stirling numbers.** (For a detailed study of this matter, see [Moser, Wyman, 1958b, c].) We suppose \(k\) and \(a\) fixed, and \(n \to \infty\). (1) \(S(n, k) \sim k^n/k!\). [Hunt: [7d], p. 194, and [1b], p. 204.] (2) \(s(n+1, k+1) \sim (n/k!)^k\log n\). Moreover, [7b] (p. 217) gives a complete asymptotic expansion.

*10. **Alike binomial coefficients (ABC).** These are integers of the form \(n!(a! b!)^{-1}\), where \(n, a, b\) are integers too, such that \(a + b \geq n\). Every binomial is evidently \(ABC\). Show the existence of a universal constant \(c > 0\) such that \(a + b < n + c \log n\) for each \(ABC\) ([Erdos, 1968]).

*11. **Around a definition of \(e\).** It is well known that \(e(t) = (1 - t)^{-1}\) approaches \(e\) if \(t\) tends to 0. More precisely, \(e(t) = e(1 + \sum_{n \geq 1} A(n)t^n) = -\sum_{n \geq 1} a(n)t^n\), where the rational numbers \(A(1), A(2), \ldots\) equal \(1/2, 11/24, 7/16, 2447/5760, 959/2304, 238043/580608, 67223/168888\), and where \(a(n) = eA(n)\) has an asymptotic expansion:

\[a_n \approx 1 + \frac{1}{n} + \sum_{v \geq 1} P_v(\log n)\frac{1}{n^v}\]

where \(n \to \infty\) and \(P_v(x)\) are polynomials of degree \(v\), \(P_0(x) = 1\) for \(v = 1\), \(-1, -x, \ldots\).

*12. **Inverting the harmonic numbers.** Let us consider a strictly increasing real sequence \(f(n), n \geq a, b = f(a), f(\infty) = \infty\). For any real number \(x \geq b\), we write \(f^{-1}(x)\) for the largest integer \(n \leq x\). For example, if \(f(n) = n\), we find \(f^{-1}(x) = \lfloor x \rfloor\), the integral part of \(x\). (1) For the harmonic sequence \(f(n) = 1 + 2^{-1} + 3^{-1} + \cdots + n^{-1}\) and for any \(x \geq 2\), we have \(f^{-1}(x) = \lfloor e^{\gamma} - (1/2) - (3/2)(e^{\gamma} - 1) - 1\rfloor\) or the same integer plus one ([Comtet, 1967], [Boas, Wrench, 1971]. \(\gamma = 0.5772\ldots\) is the Euler constant). (2) More generally, calculate \(f^{-1}(x)\), where \(f(n) = 1 + 2^{-s} + 3^{-s} + \cdots + n^{-s}\), \(s < 1\).

*13. **Cauchy numbers (or integral of the rising and falling factorials).** (See [Liénard, 1946], [Nyström, 1930], [Wachs, 1947]). Let us consider the
Cauchy numbers of the first type \(a_n := \int_0^1 x^n \log(1 + x) \, dx\) and of the second type \(b_n := \int_0^1 x^n \log(1 - x) \, dx\). (1) \(\sum a_n x^n/n! = \exp((1 + x)/2) - 1\), \(\sum b_n x^n/n! = (-1/2) \exp((1 - x)/2) - 1\). (2) \(a_n = \sum s(n, k)/(k + 1)\), \(b_n = \sum s(n, k)/(k + 1)\). (3) \(a_n = \sum_{k=0}^n (-1)^{n-k} a_{n-k}(k+1)\), \(b_n = \sum_{k=0}^n (-1)^{n-k} b_{n-k}(k+1)\).

\[
\begin{array}{c|cccccccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 \hline
 a_n & 1 & 1 & 6 & 4 & 30 & 4 & 24 & 90 & 20 & 132 \\
 b_n & 1 & 5 & 9 & 251 & 475 & 19087 & 36799 & 10700167 & 2082753 & 134211265 \\
\end{array}
\]

(4) When \(n \to \infty\), we have \(a_n/n! \sim (-1)^{n+1} n^{-1} (\log n)^{-2}\) and \(b_n/n! \sim (-1)^{n+1} n^{-1} (\log n)^{-2}\).

14. Representations of zero as a sum of different summands between \(n\) and \(-n\). Let \(A(n)\) be the number of solutions of \(x_i = -1\), \(k x_i = 0\), where \(x_i\) equals 0 or 1. Then, when \(n \to \infty\), \(A(n) \sim n^{1/2} \log^{1/2} n^{-3/2}\) ([Van Lint, 1967], [Entringer, 1968]).

15. Sum of the inverses of binomial and multinomial coefficients. The sequence \(L_n := \sum_{k=0}^n \binom{n}{k}^{-1}\) equals \((n + 1)2^{-n-1} \sum x_k = 1 \leq k\). (For a probabilistic remark, see [*Letac, 1970], p. 14). It satisfies the recurrence \(L_n = ((n + 1)/2n) L_{n-1} + 2\) and has the following (divergent) asymptotic expansion: \(L_n/2 = 1 + \sum_{p \geq 0} b_p n^{-p-1}\), where the integers \(b_p\) have as GF: \(\sum_{p \geq 0} b_p x^p/p! = (2 - e^x)^{-2}\).

\[
\begin{array}{c|cccccccccc}
p & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
 b_p & 1 & 2 & 8 & 30 & 2612 & 25988 & 296564 & 2816548 & 56667412 & 862584068 \\
\end{array}
\]

In the same way, \(L_n(x) := \sum_{k=0}^n \binom{n}{k}^{-1} x^k = (n + 1)(1 + x) n^{-1} x^{-1}\) and \((1 + (1/x)) L_n(x) = (1 + 1/n) I_{n-1}(x) + x^n + x^{-1} + x^{-1} (1 + x)^n (1 + x) n^{-1} x^{-1}\).

*16. The coefficients of \(\sum n! x^n\) ([Comtet, 1972]). Let \(e(t)\) be the purely formal series \(\sum_{n \geq 0} n! t^n\). We define the coefficients \(f(n)\) by \(e(t)^{-1} = 1 - \sum_{n \geq 1} f(n) t^n\). (1) The \(f(n)\) are positive integers such that \(f(p+1) = 1 \mod p\) for \(p\) prime. (2) We have the following asymptotic expansion \(f(n)/n! \approx 1 - \sum_{k \geq 1} A_k(n)(n) = 1 - \sum_{k \geq 1} A_k(n)(n) = 1 - 2/n - 4/(n^2) - 6/(n^3) - \ldots\), where \(A_k = \sum_{j=0}^k A_k(j+1) (k+1)\). (3) The sequence \(f(n)/n!\) (which tends to 1) is increasing for \(n \geq 2\) and concave for \(n \geq 4\). (Cf. Exercise 14, p. 261 and Exercise 15, p. 294.)

*17. Sum of the logarithms of the binomial coefficients. (1) Show that \(\lim_{n \to \infty} \left\{ \sum_{k=0}^n \binom{n}{k} \log \binom{n}{k} \right\} = 1/2\). (2) More generally, for all integers \(p \geq 1\), we have \(\lim_{n \to \infty} \left\{ \sum_{k=0}^n \binom{n}{k} \log \binom{n}{k} \right\} = p/2\). ([Gould, 1964b], p. 14). (For a generalization, [Carlitz, 1966a], [Hayes, 1966b].)

*18. Examples of applications of the method of Darboux (p. 277). Determine the asymptotic expansions for the Bernoulli and Euler numbers (p. 48), the \(c_n\) (p. 56).

*19. \(r\)-orbits of a random permutation. In the probability space defined on p. 279, for each integer \(r \geq 1\), we introduce the RV \(C_r\), equal to the number of \(r\)-orbits of \(\sigma\). Show that the GF for its probabilities equals \(\sum_{k \leq r \leq n} \binom{n}{r} (r-1)!/l!\). Deduce that, for \(r\) fixed, and \(n\) tending to \(\infty\), \(C_{n,r}\) "tends" to \(\) Poisson RV with parameter \(1/r\).

*20. The number of orbits in a random derangement. We define the associated Stirling numbers of the first kind \(s(n, k)\) by \(\sum_{k=0}^n s(n, k) t^k/n! = e^{-1/(1 + t)}\). (1) The number \(d(n, k)\) of derangements of \(N, |N| = n\), with \(k\) orbits (p. 231, and Exercise 7, p. 256) equals \(s(n, k)\). (2) The polynomials \(P_n(u) := \sum_{k=0}^n d(n, k) u^k\) have all different and nonpositive roots ([Tricomi, 1951], [Carlitz, 1958a]). (3) We consider the 'random' derangements \(\sigma\) of \(N\) (for which we must specify the probability space!), and the RV \(A_n = A_n(\sigma)\) = the number of orbits of \(\sigma\). Study the asymptotic properties of the \(d(n, k)\), analogous to those of \(s(n, k)\) (pp. 279–283).
**21. Random partitions of integers.** We consider all partitions \( \omega \) of \( n \) equally probable, \( P(\omega) = (p(n))^{-1} \). We let \( S_\omega \) be the RV equal to the number of summands of \( \omega \). Hence \( P(S_\omega = m) = P(n, m) / p(n) \) (p. 94). Show that \( E(S_\omega) = (p(n))^{-1} \sum r d(r) p(n - r) \), where \( d(r) \) is the number of divisors of \( r \). [Hint: Take the derivative with respect to \( u \), (p. 97), put \( u = 1 \), and use Exercise 16 (p. 161).] (For estimates of the first three moments, see [Lutra, 1958], and for the abscissa of the ‘peak’, [Szekeres, 1953].)

**22. Random tournaments.** We define a random tournament (cf. p. 68) \( T = T(\omega) \) over \( N, |N| = n \), by making random choices for each pair \( \{x_i, x_j\} \in P_2(N) \), the arcs \( x_i \to x_j \) and \( x_j \to x_i \) being equiprobable, and the \( \binom{n}{2} \) choices independent. (1) Let \( C_n = C_n(\omega) \) be the number of 3-cycles of \( T(\omega) \) (for example, \( \{x_1, x_2, x_3\}, \) Figure 18, p. 68, is a 3-cycle. Show that \( E(C_n) = (1/4) \binom{n}{3} \), var \( C_n = (3/16) \binom{n}{3} \). [Hint: Define \( n \) random variables \( X_{i,j,k} \in P_3[N] \), by \( X_{i,j,k} = 1 \) if \( \{x_i, x_j, x_k\} \) is the support of a 3-cycle, and \( := 0 \) otherwise; observe then that \( E(X_{i,j,k}) = 1/4 \).] (2) More generally, let \( C_n \) be the number of \( k \)-cycles of \( T(\omega) \), then we have \( E(C_n) = \binom{n}{k}(k-1)!2^{-k} \) and \( \text{var} C_n = O(n^{2k-3}) \) when \( n \to \infty \). (A deep study and a vast bibliography on random tournaments are found in [Moon, 1968].)

**23. Random partitions of a set.** mode of \( S(n, k) \). With every finite set \( N, |N| = n \), we associate the probability space \( (Q, \mathcal{A}, P) \), where \( Q \) is the set of partitions of \( N \), \( \mathcal{A} = \mathcal{P}(Q) \), and \( P(\omega) = 1/|\mathcal{P}(\omega)| = 1/m(n) \) (p. 210) for each partition \( \omega \in \mathcal{P} \). We are now interested in the study of the RV \( B_n = B_n(\omega) \) for \( n \to \infty \). (1) Put \( \mathcal{P}(\omega)/m(n) = S(n, k)/w(n) \), where \( s(n, k) \) is the Stirling number of the second kind (p. 204). The GF of the probabilities is hence equal to \( P(u) = \sum S(n, k)u^k \). (2) The moments \( \mu_n \) (not central) of \( B_n \) equal:

\[
\sum_{i=0}^{n} \frac{m(n + i)}{\omega(n)} \sum_{i \leq j \leq m} \frac{(-1)^{k-i}}{j!} S(m, k) s(j, i).
\]

**24. Random words.** Let \( \mathcal{X} := \{x_1, x_2, \ldots, x_n\} \) be a finite set, or alphabet, \( |\mathcal{X}| = n \). At every epoch \( t = 1, 2, 3, \ldots \), we choose at random a letter from \( \mathcal{X} \), each letter having the probability \( 1/n \), and the choices at different moments are independent. In this way we obtain an infinite random word \( f \), and the section consisting of the first \( t \) letters is called \( f(t) \). In the sequel of this text, \( T = T(\mathcal{X}) \) is the RV which equals the first epoch that a certain event \( \mathcal{E} \) concerning \( f \) occurs. (1) Birthdays. \( \mathcal{E} \) is the event “one of the letters of \( f(t) \) has appeared \( k \) times”. Put \( \exp u = 1 + u + u^2/2! + \cdots + u^k/k! \), and show that:

\[
E(T) = \int_0^\infty \{\exp_{k-1}(t/n)\}^n e^{-t} dt.
\]

Use this to obtain, for fixed \( k \), \( E(T) \sim (k!)^{1/k} (1 + 1/k) \cdot n^{-1/k} \) for \( n \to \infty \). (For \( n = 365, k = 2 \), one needs in average 23 guests to a party, to find that two of them have the same birthday, which may be surprising.) (2) The matchboxes of Banach. A certain mathematician always carries two matchboxes with him. Both contain initially \( k \) matches. Each time he wants a match, he draws a box at random. Certainly a moment will come that he draws an empty box. Let \( R \) be the RV equal to the number of matches left in the other box. Show
that $P(R=r) = \left(\frac{2k-r}{k}\right)^r \left(\frac{2k}{k}\right)^{2-2k}$ and that $E(R) = (2k+1)2^{2k} \left(\frac{2k}{k}\right) - 1 \approx 2\sqrt{\frac{k}{\pi}} - 1$ ([*Feller, 1968]*1, p. 238, [*Kac, 1962]*). Also compute the moments $E(R^m)$, $m = 2, 3, \ldots$. (3) **Picture collector.** $E$ is the event "each letter has appeared $k$ times" in $f(t)$. Then $E(T) = \log n + (k-1)n \log n$ and $\log n + (k-1) \log (k-1) \log n(n-\infty, \gamma = \text{the Euler constant}. ([*Newman, Shepp, 1960]*, [*Erdős, Rényi, 1961]*). (Thus, when every bar of chocolate goes together with a picture, one must buy in average $\frac{1}{\log \log n + n(y - \log (k-1)!)} + o(1)$ when $n \to \infty$. $y$ is the Euler constant.

If the $I$ letters of $g$ are different, $1 < I < n$ of these bars in order to obtain the complete collection of different pictures used by the manufacturer.) (4) **The monkey typist.** Let $g$ be a word of length $n$ and $S$ the event "the last 2 letters off(t) form the word $g$".

*25. Similarly loaded dice. (1) Show that it is not possible to load similarly two dice in such a way that the total score will be an equidistributed RV (on the values $2, 3, \ldots, 12$). [*Hint: In the contrary case, use the GF of the probabilities to show the existence of $x_0, x_1, \ldots, x_n$ such that $(x_0 + x_1 + x_2 + \cdots + x_n)^n = K(t^2 + t^3 + \cdots + t^{11} + t^{12})$] (2) The following is a more difficult question ([*Clements, 1968]*) suggested by the preceding. Let $x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ and $r$ an integer $\geq 1$. We define the $c_r(x)$ by: $(x_0 + x_1 + x_2 + \cdots + x_n)^r = \sum_{\sigma=0}^n c_r(x)\sigma$ and put $M(x) = \max_{0 \leq n \leq rm} c_r(x)$ for the set of all $x$ such that $x_0, x_1, \ldots, x_n \geq 1$ and $x_0 + x_1 + \cdots + x_n = 1$. (3) Answer these two questions when the two dice may be independently loaded.

*26. Multicolour Ramsey numbers. Let be given integers $b, p_1, p_2, \ldots, p_k$ such that $1 \leq b < p_1 < p_2 < \cdots < p_k$. A finite set $N$ is called *Ramsey-*($b; p_1, p_2, \ldots, p_k$) if and only if, for all $k$-divisions $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k$ of $\mathcal{P}_k(N)$, $\mathcal{P}_k(N) = \mathcal{C}_1 + \mathcal{C}_2 + \cdots + \mathcal{C}_k$, there exists an integer $i \in [k]$ and a block $P \in \mathcal{P}_k(N)$ such that $P \in \mathcal{C}_i$. (1) Show by induction on $k$, the existence of $k$-color $b$-ary Ramsey numbers, denoted by $\rho(b; p_1, p_2, \ldots, p_k)$ and satisfying: $N$ is Ramsey-($b; p_1, p_2, \ldots, p_k$) $\iff |N| \geq \rho(b; p_1, p_2, \ldots, p_k)$.

(2) Moreover, show that $\rho(1; p_1, p_2, \ldots, p_k) = p_1 + p_2 + \cdots + p_k - k + 1$ ([*Ryser, 1963]*, p. 39, and [*Dembowski, 1965]*, p. 29). We note: $\rho(2; 3, 3, 3) = 17, \rho(2; 3, 3, 4) \geq 30, \rho(2; 4, 4, 4) \geq 80, \rho(2; 5, 5, 5) \geq 102, \rho(2; 3, 3, 3, 3, 3) \geq 278$. (3) As an application of (1) show that for every integer $k \geq 1$ there exists an integer $R(k)$ with the following property: when $n \geq B(k)$, each $k$-division $(A_1, A_2, \ldots, A_k)$ of $[n]$, $A_1 + A_2 + \cdots + A_k = [n]$, is such that one of the subsets $A_i$ contains three numbers of the form $x, y, x+y$. [*Hint: For $n \geq \rho(2; 3, 3, ..., 3)$, where the number 3 occurs $k$ times, apply (1) to the division $\mathcal{C}_1 + \mathcal{C}_2 + \cdots + \mathcal{C}_k = \mathcal{P}_k(n)$ defined by: $(a, b) \in \mathcal{C}_i \iff a - b \in A_i$.]

*27. Convex polygons whose vertices form a subset of a given point set of the plane ([*Erdős, Szekeres, 1935]*, explained in [*Ryser, 1963]*, p. 43, and [*Dembowski, 1965]*, p. 30). Let $N$ be a finite set of points in the plane such that no three among them are collinear, $N$ is general, for short. An $m$-gon extracted from $N$ will be the following: a closed polygonal line $\mathcal{P}$, not necessarily convex, whose vertices are different and belong to $N$. Such a polygon $\mathcal{P}$ is considered as a set of pairs of $N$ (its sides), $\mathcal{P} \subset \mathcal{P}_2(N)$. (1) From every general set $A, |A| = 5$, we can extract a convex quadrilateral. (2) Let $M$ be a general set, $|M| \geq 4$, such that for each $Q \subset M$, $|Q| = 4$, one of the three quadrilaterals whose vertex set is $Q$ is convex. Then, there exists a convex $m$-gon extracted from $M, |M| = m$. [*Hint: If not, the convex hull of $M$ would be spanned by less than $m$ points, consequently there would exist a $Q$ whose three quadrilaterals are not convex.] (3) Deduce from (1, 2) the following theorem: For every integer $m \geq 20$ there exists an integer $f(m)$ such that from every finite general set containing at least $f(m)$ points of the plane, a convex $m$-gon can be extracted. [*Hint: We have $f(3) = 0, f(4) = 4$ for $m \geq 5$, apply the theorem on p. 283, $p \to m, q \to 5, \mathcal{C} = \mathcal{P}_4(N)$, where $\mathcal{C}$ is the set of the $Q, |Q| = 4$, such that one of 3 extracted quadrilaterals is convex.]

28. Monotonic subsets of a sequence. Let $X$ be a set of real numbers $>0$, $X := \{x_1, x_2, x_3, \ldots\}, 0 < x_1 < x_2 < x_3 < \cdots$. For all integers $h, k \geq 1$, we put $r(h, k) := (h-1)(k-1)+1$. Let $N$ be a subset with $n$ elements of $X$, $N \subset X, |N| = n$, and let $\psi$ be a map of $N$ into $R$. We first suppose that $n = r(h, k)$. Show that there exists a subset $H \subset N, H = |H|$ on which the restriction of $\psi$ is increasing (not necessarily strictly), or a subset $K \subset N, |K| = k$, on which $\psi$ is decreasing (not necessarily strictly). [*Hint: Argue by induction on $k \geq 2$, and fixed $h$. For $A \subset N$, $|A| = r(h, k) - 1,$
30. **Complete subgraphs in graphs with sufficiently high degrees.** A necessary and sufficient condition that every graph \( S \) of \( |N| = n, \) all of its degrees exceeding or equalling \( k \) \((\forall x \in N, |S(x)| \geq k, \) p. 61\), contains a complete subgraph with \( p \) nodes is \( k > n(p-2)/(p-1) \) ([Turán, 1941]. [Zarankiewicz, 1947]).

31. **Maximum of a certain quadratic form** ([Motzkin, Straus, 1965]). Let \( E \) be the set of vectors \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) whose real coordinates \( x_i \) satisfy \( x_i \geq 0, \) \( i \in [n], \) and \( x_1 + x_2 + \cdots + x_n = 1. \) (1) Let \( F(x) \) denote the quadratic form \( \sum_{i,j \in \Phi} x_i x_j \) (for instance \( F_3 = x_1 x_2 + x_2 x_3 + x_3 x_1 \)). Show that \( \max_{x \in E} F(x) = \left(1 - \frac{1}{n}\right)/2. \) (2) More generally, let \( G \) be a graph over \( [n] = \{1, 2, \ldots, n\} \) (p. 61) and \( F_G(x) = \sum_{i,j \in \Delta} x_i x_j. \) Show that \( \max_{x \in E} F_G(E) \) equals \( \left(1 - \frac{1}{k}\right)/2, \) where \( k \) is the maximum number of nodes of complete subgraphs contained in \( G \) (p. 62), in other words, the maximum value of the number of elements of sets \( H \subset [n] \) such that \( \Psi_S(H) \in E. \)

**Hint:** If \( K = \{i_1, i_2, \ldots, i_k\} \) is the set of nodes of a complete subgraph of \( G, \) then a lower bound for \( \max F_G(x) \) is given by \( F_G(x) \) for \( x_j = 1/k \) if \( i \in K \) and \( = 0 \) otherwise. For the other inequality, use induction; first observe that the maximum occurs in an interior point of \( E. \)

32. **Systems of distinct representatives.** Let \( \mathcal{S} = \{B_1, B_2, \ldots, B_m\} \) be a system of blocks, not necessarily different from \( N, B_i \subset N, \) \( i \in [m]. \) A block \( M = \{x_1, x_2, \ldots, x_m\} \subset N \) is called a system of distinct representatives, abbreviated SDR, if and only if \( x_i \in B_i \) for all \( i \in [m]. \) A necessary and sufficient condition that \( \mathcal{S} \) admits a SDR is that for every subsystem \( \mathcal{S}' \subset \mathcal{S} \) we have \( |\cup_{B \in \mathcal{S}'} B| \geq |\mathcal{S}'|. \) (The preceding statement, due to [Hall, 1935], answers in particular the 'marriage problem': \( m \) boys know a certain number of girls; under what conditions can each boy marry a girl he knows already? One girl may be acquainted with several boys!...) See also [Halmos, Vaughan, 1960], [Mirsky, Perfect, 1966].

33. **Agglutinating systems.** A system \( \mathcal{S} \) of blocks of \( N, \mathcal{S} \subset \mathcal{B}'(N), \) \( |N| = n, \) is called agglutinating if any two of them are not disjoint. Show that the number \( |\mathcal{S}'| \) of blocks of such a system is less than or equal to \( 2^{n-1}, \) and that this number is a least upper bound. **Hint:** Use the Cayley table (= the multiplication table) of \( G \) to show that \( g(n) \leq n^2. \) (2) The Cayley table of \( G \) is completely known if we know it for \( \Delta \times G \) only, where \( \Delta \) is a system of generators of \( G \) (3) Let \( G \) be a minimal system of generators (\( \exists \) there does not exist a system of generators with a smaller number of elements). Show that \( 2^{\log_2 n} \leq n. \) Deduce that \( g(n) \leq n^{\log_2 n}, \) where \( \log_2 n \) means the logarithm with base 2 of \( n. \) ([Gallagher, 1967]. The following table of
g(n) is taken from [*Coxeter, Moser, 1965], p. 134. See also [Newman, 1967], [James, Connor, 1969].

30. Multicoverings. An l-multicovering of N is any system \( \mathcal{G} \subseteq \mathcal{P}(N) \) such that each \( x \in N \) is contained in exactly \( l \) blocks of \( \mathcal{G} \); (the blocks are all different). Compute and estimate the number of l-coverings of \( N, |N| = n \) ([Comtet, 1968b], [Baroti, 1970], [Rényi, 1971, p. 30]). Here are the first values of \( C_2(n, k) \), the number of bicoverings of \( N \) with \( k \) blocks, and \( C_2(n) := \sum_k C_2(n, k) \), the total number of bicoverings:

<table>
<thead>
<tr>
<th>n</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>g(n)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>( C_2(n) )</td>
<td>1</td>
<td>4</td>
<td>4</td>
<td>8</td>
<td>13</td>
<td>39</td>
<td>25</td>
<td>3</td>
</tr>
</tbody>
</table>

41. At most \( 1 \)-overlapping systems. These are systems \( \mathcal{G} \) of \( N \) consisting of \( k \) blocks, \( \mathcal{G} \subseteq \mathcal{P}(N) \), such that for any \( A \) and \( B, A \neq B \), we have \( |A \cap B| \leq 1 \). If \( \varphi(n, k) \) is the largest possible number of blocks of such a system \( \mathcal{G} \), show that \( \varphi(n, k) \sim n^2/k(k-1) \), for \( n \to \infty \) ([Erdős, Hanani, 1963]).

42. Geometries. A geometry (or linear system) of \( N \) is a system \( \mathcal{G} \subseteq \mathcal{P}(N) \) whose blocks, called lines, satisfy the following two conditions: (1) Each pair \( A \subseteq N, |A| = 2 \), is contained in precisely one line; (2) each line contains at least two points. The following are the known values of \( \varphi(n) \), which is the number of geometries of \( N, |N| = n \), and the numbers \( \varphi^*(n) \), which are the number of nonisomorphic ones:

<table>
<thead>
<tr>
<th>n</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varphi(n) )</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>32</td>
<td>353</td>
<td>8390</td>
<td>436399</td>
<td>50468754</td>
</tr>
<tr>
<td>( \varphi^*(n) )</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td>10</td>
<td>24</td>
<td>64</td>
<td>384</td>
<td></td>
</tr>
</tbody>
</table>

Compute and estimate \( \varphi(n) \) and \( \varphi^*(n) \) (for \( n \geq 10 \), we have \( 2^n < \varphi^*(n) < \varphi(n) < 2^{(n)} \)) these inequalities and their numerical values being due to [Doyen, 1967].

43. Steiner triple systems. A Steiner triple system over \( N \) or simply a 'triple system', is a set \( \mathcal{G} \) of triples of \( N, \mathcal{G} \subseteq \mathcal{P}_3(N) \), such that every pair
of elements of $N$ is contained in exactly one triple. In the sense of the previous exercise, this is a 'geometry' in which every line has three points.

We suppose $N$ finite, $|N| = n$. (1) A necessary and sufficient condition for the existence of a triple system is that $n$ is of the form $6k + 1$ or $6k + 3$.

(2) Let $s(n)$ denote the number of triple systems (of $N$), and $s^*(n)$ the number of nonisomorphic ones. The known values are:

Computing and estimating $s(n)$ and $s^*(n)$, where $n \equiv 1$ or $3 \pmod{6}$. (See [Doyen, Valette, 1971].)

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>3</th>
<th>7</th>
<th>9</th>
<th>13</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s(n)$</td>
<td>1</td>
<td>1</td>
<td>30</td>
<td>840</td>
<td>1197504000</td>
<td>60281712691200</td>
</tr>
<tr>
<td>$s^*(n)$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>80</td>
</tr>
</tbody>
</table>
The number $P(n, m)$ of partitions of $n$ into $m$ summands and the number $p(n) = \sum_{m} P(n, m)$ of partitions of $n$:

<table>
<thead>
<tr>
<th>$p(n)$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
</tr>
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For $m \geq n/2$ (right of the bold-face figures), the table is completed by $P(n, m) = -p(n - m)$. A table of $P(n, m)$, $n \leq m \leq 100$, is found in [Todd, 1944].
Stirling numbers of the first kind \( s(n, k) \)

\[
\begin{array}{cccccc}
 n \backslash k & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 2 & 3 & 4 & 5 & 6 & 1 \\
4 & 11 & 16 & 21 & 26 & 31 & 1 \\
5 & 65 & 125 & 210 & 290 & 370 & 450 \\
6 & 401 & 1050 & 2100 & 3150 & 4200 & 5250 \\
7 & 3003 & 8501 & 17016 & 28030 & 39876 & 52800 \\
8 & 23016 & 66516 & 133032 & 209296 & 285096 & 360768 \\
9 & 178160 & 535320 & 1070720 & 1716320 & 2361920 & 3007520 \\
10 & 1551120 & 4657600 & 8735200 & 13323200 & 18304800 & 23583840 \\
11 & 14542236 & 46010440 & 86151000 & 134555500 & 187075400 & 243876840 \\
12 & 148387876 & 491614200 & 890228800 & 1378188800 & 1887504000 & 2508107840 \\
13 & 1558982636 & 5350661280 & 9438242880 & 14722649600 & 20592403840 & 27354451840 \\
14 & 16795042638 & 58781899840 & 105277679040 & 163987835200 & 233120748800 & 313845223040 \\
15 & 184259745906 & 652223469440 & 1151479763840 & 18149322067200 & 26068982003200 & 35223334592000 \\
\end{array}
\]

For a table of the \( s(n, k) \), \( k \leq 60 \), see [Mitrinović (D. S. and R. S.), 1960a, b, 1961] and for several extensions

Stirling numbers of the second kind \( S(n, k) \) and exponential numbers \( \omega(n) = \sum S(n, k) \)

\[
\begin{array}{cccccc}
 n \backslash k & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 3 & 1 & 1 & 1 & 1 \\
4 & 1 & 7 & 6 & 1 & 1 & 1 \\
5 & 1 & 15 & 25 & 10 & 1 & 1 \\
6 & 1 & 31 & 90 & 65 & 15 & 1 \\
7 & 1 & 63 & 301 & 350 & 140 & 21 \\
8 & 1 & 127 & 966 & 1701 & 1050 & 266 \\
9 & 1 & 255 & 3025 & 7770 & 6951 & 2646 \\
10 & 1 & 511 & 9330 & 34105 & 42525 & 22827 \\
11 & 1 & 1023 & 28501 & 145750 & 246730 & 179487 \\
12 & 1 & 2047 & 86526 & 611501 & 1379400 & 1332652 \\
13 & 1 & 4095 & 261625 & 2532530 & 7508501 & 9321312 \\
14 & 1 & 8191 & 788970 & 10391745 & 40075035 & 63463673 \\
15 & 1 & 16383 & 2375101 & 42355950 & 210766920 & 420693273 \\
\end{array}
\]

For a table of \( S(n, k) \), \( k \leq 27 \), see [Miksa, 1956], and for \( \omega(n) \), \( n \leq 74 \) [Levine, Dalton, 1962].

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